Bootstrap Methods: Recent Advances and New Applications

2007 ICTP Lectures

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Bootstrap Topics

- Introduction to Bootstrap
- Wide Variety of Applications
- Confidence regions and hypothesis tests
- Examples where bootstrap is not consistent and remedies:

(1) infinite variance case for a population mean and
(2) extreme values

- Available Software

Introduction

- The bootstrap is a general method for doing statistical analysis without making strong parametric assumptions.
- Efron's nonparametric bootstrap, resamples the original data.
- It was originally designed to estimate bias and standard errors for statistical estimates much like the jackknife.

Introduction (continued)

- The bootstrap is similar to earlier techniques which are also called resampling methods:
  - (1) jackknife,
  - (2) cross-validation,
  - (3) delta method,
  - (4) permutation methods, and
  - (5) subsampling..

Introduction (continued)

The technique was extended, modified and refined to handle a wide variety of problems including:

- (1) confidence intervals and hypothesis tests,
- (2) linear and nonlinear regression,
- (3) time series analysis and other problems

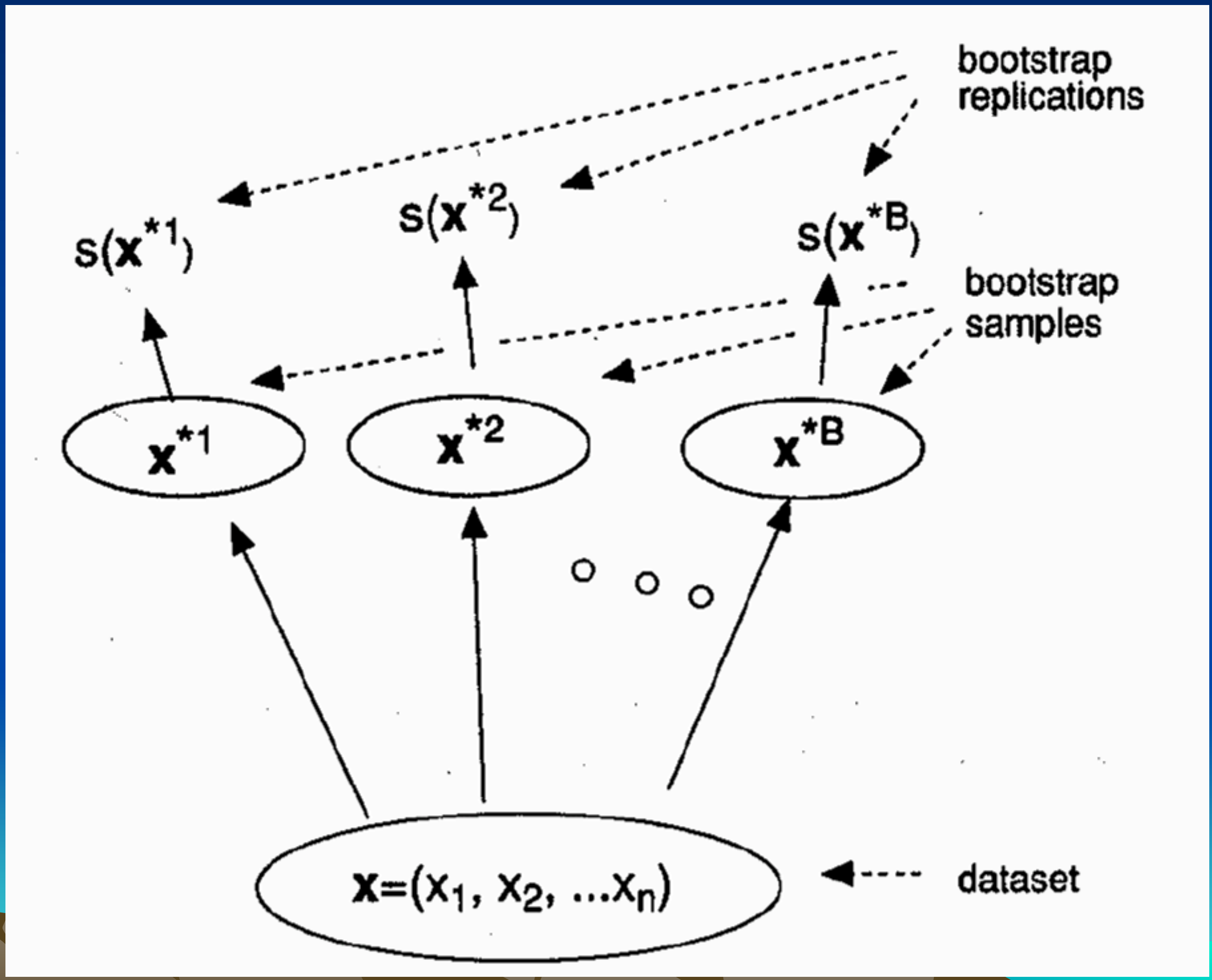
Introduction (continued)

Definition of Efron's nonparametric bootstrap. Given a sample of n independent identically distributed (i.i.d.) observations $X_1, X_2, \ldots, X_n$ from a distribution F and a parameter θ of the distribution F with a real valued estimator $\theta(X_1, X_2, \ldots, X_n)$, the bootstrap estimates the accuracy of the estimator by replacing F with F_n, the empirical distribution, where F_n places probability mass 1/n at each observation X_i.

Introduction (continued)

- Let $X_1^*, X_2^*, \ldots, X_n^*$ be a bootstrap sample, that is a sample of size n taken with replacement from F_n .
- The bootstrap, estimates the variance of $\theta(X_1, X_2, \ldots, X_n)$ by computing or approximating the variance of $\theta^* = \theta(X_1^*, X_2^*, \ldots, X_n^*)$.

Schematic of Bootstrap Process



Introduction (continued)

- Statistical Functionals - A functional is a mapping that takes functions into real numbers.
- Parameters of a distribution can usually be expressed as functionals of the population distribution.
- Often the standard estimate of a parameter is the same functional applied to the empirical distribution.

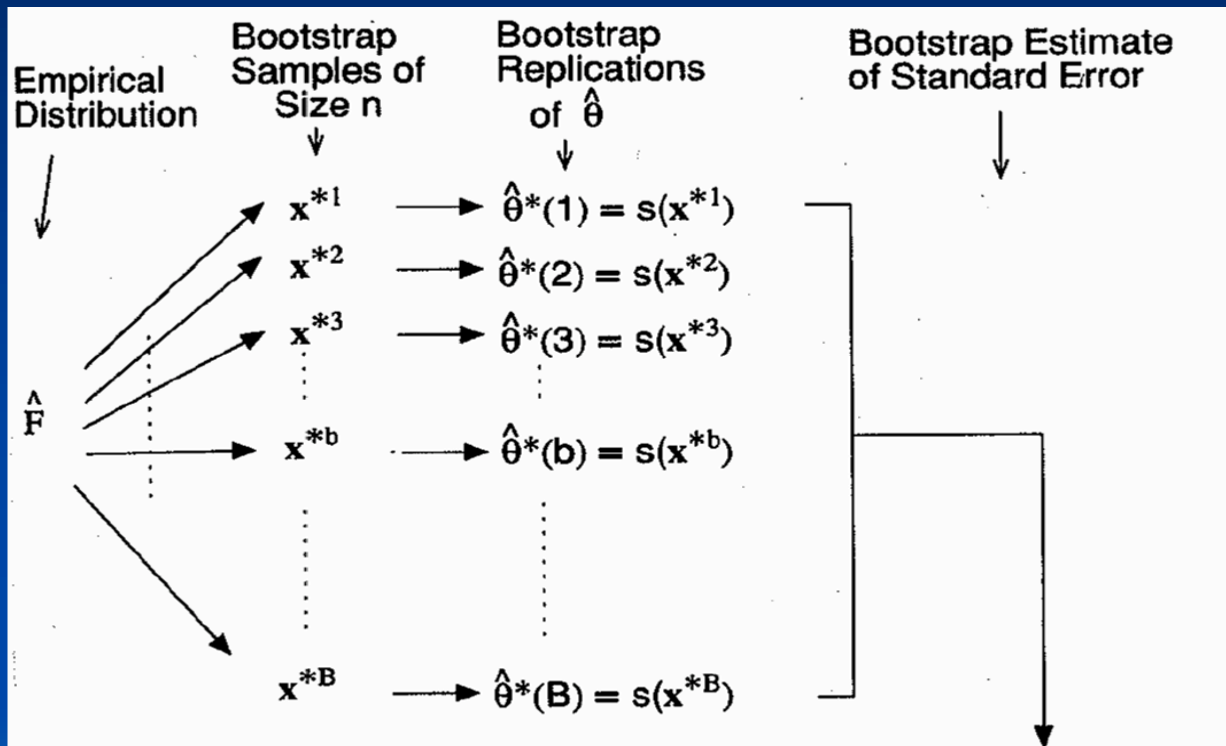
Introduction (continued)

- Statistical Functionals and the bootstrap.
- A parameter θ is a functional T(F) where T denotes the functional and F is a population distribution.
- An estimator of θ is $\theta_h = T(F_n)$ where F_n is the empirical distribution function.
- Many statistical problems involve properties of the distribution of $\theta - \theta_h$, its mean (bias of θ_h), variance, median etc.

Introduction (continued)

- Bootstrap idea: Cannot determine the distribution of $\theta - \theta_h$ but through the bootstrap we can determine, or approximate through Monte Carlo, the distribution of $\theta_h - \theta^*$, where $\theta^* = T(F_n^*)$ and F_n^* is the empirical distribution for a bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ (θ^* is a bootstrap estimate of θ).
- Based on k bootstrap samples the Monte Carlo approximation to the distribution of $\theta_h - \theta^*$ is used to estimate bias, variance etc. for θ_h .
- In bootstrapping θ_h substitutes for θ and θ^* substitutes for θ_h . Called the bootstrap principle.

Introduction (continued)



$$\widehat{se}_B = \left[\Sigma_{b=1}^{B} \frac{[\hat{\theta}^*(b) - \hat{\theta}^*(\cdot)]^2}{B-1} \right]^{1/2}$$

where $\hat{\theta}^*(\cdot) = \Sigma_{b=1}^{B} \frac{\hat{\theta}^*(b)}{B}$

Introduction (continued)

- Basic Theory: Mathematical results show that bootstrap estimates are consistent in particular cases.
- Basic Idea: Empirical distributions behave in large samples like population distributions. Glivenko-Cantelli Theorem tells us this.
- The smoothness condition is needed to transfer consistency to functionals of F_n, such as the estimate of the parameter θ.

Wide Variety of Applications

- Efron and others recognized that through the power of fast computing the Monte Carlo approximation could be used to extend the bootstrap to many different statistical problems .

Wide Variety of Applications (continued)

- It can estimate process capability indices for non-Gaussian data.
- It is used to adjust p-values in a variety of multiple comparison situations.
- It can be extended to problems involving dependent data including multivariate, spatial and time series data and in sampling from finite populations.

Wide Variety of Applications (continued)

- It also has been applied to problems involving missing data.
- In many cases, the theory justifying the use of bootstrap (e.g. consistency theorems) has been extended to these non i.i.d. settings.
- In other cases, the bootstrap has been modified to "make it work." The general case of confidence interval estimation is a notable example.

Confidence regions and hypothesis tests

- The percentile method and other bootstrap variations may require 1000 or more bootstrap replications to be very useful.
- The percentile method only works under special conditions.
- Bias correction and other adjustments are sometimes needed to make the bootstrap “accurate” and “correct” when the sample size n is small or moderate.

Confidence regions and hypothesis tests (continued)

- Confidence intervals are **accurate** or **nearly exact** when the stated confidence level for the intervals is approximately the long run probability that the random interval contains the "true" value of the parameter.
- Accurate confidence intervals are said to be **correct** if they are approximately the shortest length confidence intervals possible for the given confidence level.

Examples where the bootstrap fails

- Athreya (1987) shows that the bootstrap estimate of the sample mean is inconsistent when the population distribution has an infinite variance.
- Angus (1993) provides similar inconsistency results for the maximum and minimum of a sequence of independent identically distributed observations.

Bootstrap Remedies

- In the past decade many of the problems where the bootstrap is inconsistent remedies have been found by researchers to give good modified bootstrap solutions that are consistent.
- For both problems describe thus far a simple procedure called the *m-out-n* bootstrap has been shown to lead to consistent estimates .

The *m-out-of-n* Bootstrap

- This idea was proposed by Bickel and Ren (1996) for handling doubly censored data.
- Instead of sampling n times with replacement from a sample of size n they suggest to do it only m times where m is much less than n.
- To get the consistency results both m and n need to get large but at different rates. We need m=o(n). That is $m/n \to 0$ as m and n both $\to \infty$.
- This method leads to consistent bootstrap estimates in many cases where the ordinary bootstrap has problems, particularly (1) mean with infinite variance and (2) extreme value distributions.

Available Software

- Resampling Stats from Resampling Stats Inc. (provides basic bootstrap tools in easy to use software and is good as an elementary teaching tool).
- SPlus from Insightful Corporation (good for advanced bootstrap techniques such as BCa, easy to use in new Windows based version). The current module Resample is what I use in my bootstrap class at statistics.com.
- S functions provided by Tibshirani (see Appendix in Efron and Tibshirani text or visit Rob Tibshirani's web site http:/www.stat-stanford.edu/~tibs)

Available Software (continued)

- Stata has a bootstrap algorithm available that some users rave about.
- Mathworks and other examples (see Susan Holmes web page):

  http:/www-stat.stanford.edu/~susan) or contact her by email
- SAS macros are available and Proc MULTTEST does bootstrap sampling.

References on confidence intervals and hypothesis tests

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(2) Athreya, K. B. (1987). Bootstrap estimation of the mean in the infinite variance case. *Ann. Statist.* **15,** 724-731.

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