Uncapacitated Metric Facility Location Problem (UMFL).

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1 Uncapacitated Metric Facility Location Problem (UMFL).

- Let $\mathcal{L}$ be a set of locations.
- Let $\mathcal{F} \subset \mathcal{L}$ be a set of potential facility locations.
- Let $\mathcal{C} \subset \mathcal{L}$ be a set of clients (cities).
- Let $c : \mathcal{L} \times \mathcal{L} \to \mathbb{R}^+ \; \forall i, j \in \mathcal{L}$ be a distance function. Alternatively think of the function as describing the cost of assigning city j to facility i.
- Let $f : \mathcal{F} \to \mathbb{R}^+ \; \forall i \in \mathcal{F}$ be a cost function describing the cost of opening a facility at i.
- Let $\phi : \mathcal{C} \to \mathcal{F} \; \forall j \in \mathcal{C}$ be an assignment function. I.e. $\phi(j) = i$ if city j is assigned to facility i.

Problem: *Determine a set of facilities to open and an assignment of all cities to the open facilities that minimizes the total opening and distance cost.*

Notation 1.1. $f(i) = f_i$. $c(i, j) = c_{ij}$

The problem is uncapacitated as there is no bound on how many cities an open facility can serve. It is metric as c is defining a metric. This especially means:

$$\begin{aligned} c_{ij} &= c_{ji} && \forall\, i, j \in L \\ c_{ij} &\leq c_{ik} + c_{kj} && \forall\, i, j, k \in L \qquad \text{(triangle inequality)} \end{aligned}$$

2 IP Formulation.

$$\text{Let } y_i = \begin{cases} 1 & \text{if a facility is opened at } i \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Let } x_{ij} = \begin{cases} 1 & \text{if } \phi(j) = i \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Minimize } Z(x,y) = F(x,y) + C(x,y) = \sum_{i \in \mathcal{F}} f_i \cdot y_i + \sum_{\substack{j \in \mathcal{C} \\ i \in \mathcal{F}}} c_{ij} \cdot x_{ij}$$

$$s.t.$$

$$\sum_{i \in \mathcal{F}} x_{ij} = 1 \qquad \forall\, j \in \mathcal{C} \tag{1}$$

$$x_{ij} \leq y_i \qquad \forall\, i \in \mathcal{F},\ j \in \mathcal{C} \tag{2}$$

$$x_{ij}, y_i \in \{0,1\} \qquad \forall\, i \in \mathcal{F},\ j \in \mathcal{C} \tag{3}$$

The constraint (1) ensures that all cities gets assigned to a facility. The constraint (2) ensures that the assigned facilities are open. The LP-relaxation is obtained by changing (3) into:

$$x_{ij}, y_i \geq 0 \qquad \forall\, i \in \mathcal{F},\ j \in \mathcal{C}$$

the upper bound is unnecessary.

3 Approximation Algorithm for UMFL.

Algorithm Idea.

The algorithm will take an optimal solution for the LP-relaxation (x^*, y^*) and change this into a feasible solution for the IP problem. This will consist of two operations:

$$(x^*, y^*) \xrightarrow{filter} (x, y) \xrightarrow{round} (\hat{x}, \hat{y})$$

where (x, y) and $(\hat{x}, \hat{y})$ are feasible solutions to the LP- and IP-problem respectively. A high-level description of the algorithm steps are:

1. Greedily chose the city, c_{min} which ”is cheapest” i.e has the lowest overall distance cost.
2. Chose the cheapest facility location α among those “fractionally opened” locations which the city is “fractionally assigned” to.
3. Open a facility f_α at α completely.
4. Assign the city completely to the f_α facility (and only to this facility).
5. Assign all cities which are “fractionally assigned” to some facility locations in the “neighbourhood” of c_{min} completely to the the f_α facility.
6. Update collection of unassigned cities and repeat from step 1)

The reason for the filtering is step 5). If some city ”far away” is assigned with a very small $x_{ij} > 0$ value to a neighbouring facility, it will be very costly to assign this city to the newly opened facility. This is avoided by ensuring $x_{ij} = 0$ if c_{ij} is ”large”.

Filtering.

Definition 3.1. Let

$$\Delta j = \sum_{i \in \mathcal{F}} c_{ij} \cdot x_{ij} \qquad \forall\, j \in \mathcal{C}$$

Definition 3.2. $\forall\, j \in \mathcal{C}$ let $B_j = \{i \in \mathcal{F} \mid c_{ij} < 2\Delta j\}$ This describes a neighbourhood or ”ball” around each city containing facility locations with ”small” distances.

Lemma 3.3. *Given a solution (x', y') of the LP-problem, there exists a feasible solution to the LP-problem (x, y) such that:*

i) $x_{ij} > 0 \Rightarrow c_{ij} < 2\Delta j \qquad$ (*I.e.* c_{ij} *”is small”*)

ii) $Z(x, y) \leq 2\; Z(x', y')$

Proof. $\forall\, i \in \mathcal{F},\; j \in \mathcal{C}$ let

$$x_{ij} = \begin{cases} \dfrac{x'_{ij}}{\sum_{i \in B_j} x'_{ij}} & \text{if } i \in B_j \\ 0 & \text{otherwise} \end{cases}$$
$$y_i = \min\{1, 2y'_i\}$$

Observation 3.4. $x_{ij} \leq 1 \qquad \forall\, i \in \mathcal{F},\; j \in \mathcal{C}$

Claim 3.5. (x, y) *is a feasible solution fulfilling Lemma 3.3 i) .*

Check for constraint (1)

$$\sum_{i \in \mathcal{F}} x_{ij} = \sum_{i \in B_j} x_{ij} + \sum_{i \notin B_j} x_{ij} = \sum_{i \in B_j} \frac{x'_{ij}}{\sum_{i \in B_j} x'_{ij}} + \sum_{i \notin B_j} 0 = 1 + 0 = 1$$

Check for constraint (2)

Case 1. $y_i = 1$ follows from observation 3.4

Case 2. $y_i = 2y'_i$. We have $\sum_{i \in \mathcal{F}} x'_{ij} = 1$ as (x'_{ij}, y'_i) is a solution to the LP-problem. Interpret x'_{ij} as a probability distribution for “assigning j to i” and c_{ij} as a “distance” random variable.

Theorem 3.6 (Markov Inequality)**.** *Let X be a positive, random variable. Let $a > 0$ then*

$$Pr\;[\; X \geq a \;] \leq \frac{\mathbb{E}[X]}{a}$$

Using the Markov Inequality we get:

$$\sum_{i \notin B_j} x'_{ij} = Pr\ [\ c_{ij} \geq 2\Delta j\ ] \leq \frac{\Delta j}{2\Delta j} = \frac{1}{2}$$
$$\Downarrow$$
$$\sum_{i \in B_j} x'_{ij} \geq \frac{1}{2}$$
$$\Downarrow$$
$$x_{ij} \leq 2x'_{ij} \leq 2y'_i = y_i$$

The last line following from the definition of x_{ij} and from (x', y') being a feasible solution. This proves claim 3.5 and per construction part i) of Lemma 3.3

$$\begin{aligned} Z(x,y) &= F(x,y) + C(x,y) \\ F(x,y) &= \sum_{i \in \mathcal{F}} f_i \cdot y_i \leq \sum_{i \in \mathcal{F}} f_i \cdot 2y'_i = 2F(x',y') \\ C(x,y) &= \sum_{\substack{j \in \mathcal{C} \\ i \in \mathcal{F}}} c_{ij} \cdot x_{ij} \leq \sum_{\substack{j \in \mathcal{C} \\ i \in \mathcal{F}}} c_{ij} \cdot 2x'_{ij} = 2C(x',y') \\ &\Downarrow \\ Z(x,y) &\leq 2Z(x',y') \end{aligned}$$

This proves part ii) of Lemma 3.3

□

Algorithm.

Let (x', y') denote the constructed solution to the IP-problem.

Step 1. Solve the relaxed LP-problem getting optimal solution (x^*, y^*).

Step 2. Filter $(x^*, y^*) \rightarrow (x, y)$.

Step 3. Define $\Delta j = \sum_{i \in \mathcal{F}} c_{ij} x_{ij}$ and $B_j = \{i \in \mathcal{F} \mid c_{ij} < \Delta j\}$.

Observation 3.7. *No factor 2 in definition of* B_j *and* $\Delta j \leq 2 \cdot \Delta j^*,\ \forall\ j \in \mathcal{C}$

Step 4. While $\mathcal{C} \neq \emptyset$ do

- Chose minimal overall cost city:

$$j \leftarrow \min_j \Delta j$$

- Consider neighbourhood B_j. Let α be the facility location $i \in B_j$ with smallest opening cost (f_α is minimum.)
  - Open facility at α ($y'_\alpha = 1$).
  - Assign city j to α ($\phi(j) = \alpha,\ x'_{ij} = 1$ for $i = \alpha$ and $x'_{ij} = 0$ for $i \neq \alpha$)
  - Update $\mathcal{C} \leftarrow \mathcal{C} \setminus \{j\}$.
- Consider all other neighbourhoods $\overline{B_{\bar{j}}}$ for which $B_j \cap \overline{B_{\bar{j}}} \neq \emptyset \Rightarrow \exists\ \bar{i}\ \in \mathcal{F} :\ \bar{i} \in B_j$ and $\bar{i}\ \in \overline{B_{\bar{j}}}$
  - Assign city $\bar{j}$ to α ($\phi(\bar{j}) = \alpha,\ x'_{i\bar{j}} = 1$ for $i = \alpha$ and $x'_{i\bar{j}} = 0$ for $i \neq \alpha$)
  - Update $\mathcal{C} \leftarrow \mathcal{C} \setminus \{\bar{j}\}$.

Step 5. Output $\{\alpha \mid y'_\alpha = 1\}$ and ϕ.

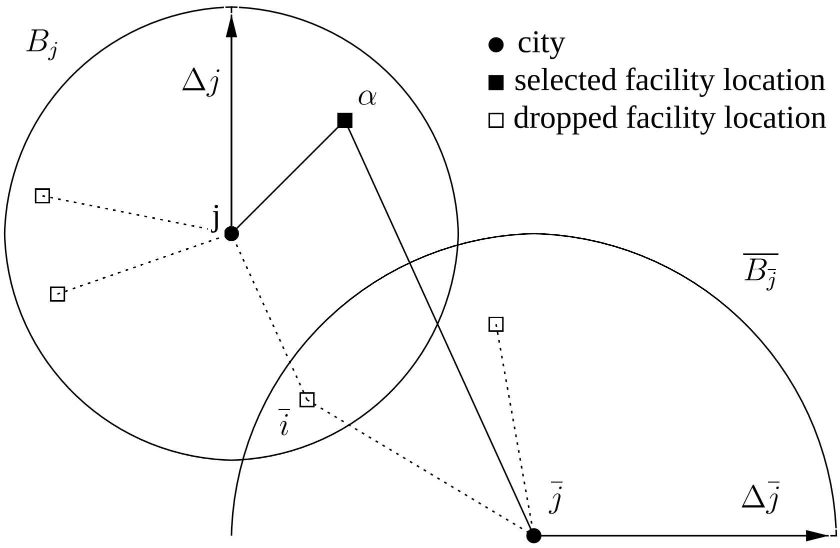

Figure 1: Assigning facility location to cities

Algorithm Analysis.

Claim 3.8. *The algorithm is a 6-approximation.*

Proof.

Termination and Feasibility: The number of cities is final and in each iteration at least one city is removed from the set of unassigned cities. The algorithm returns a feasible solution as each city has been assigned to an open facility location.

Opening Cost: Consider a round of the algorithm choosing city j.
Using the choice of α and that (x, y) is a filtered solution we have for all facility locations in B_j :

$$\sum_{i \in B_j} f_i \cdot y_i \geq \sum_{i \in B_j} f_\alpha \cdot y_i = f_\alpha \cdot \sum_{i \in B_j} y_i \geq f_\alpha \cdot \sum_{i \in B_j} x_{ij} = f_\alpha = \text{opening cost of algorithm.}$$

Let $\{\overline{B_1}, \overline{B_2}, \ldots, \overline{B_n}\}$ be all the $\overline{B_{\bar{j}}}$ sets intersecting with B_j. Define a union of disjoint sets:

$$\overline{B} = \bigcup_{i,k \in 1\ldots n} (\overline{B_i} \setminus \bigcup_{k < i} \overline{B_k})$$

We have for the facility locations in $\overline{B} \setminus B_j$:

$$\sum_{i \in \overline{B} \setminus B_j} f_i \cdot y_i \geq 0 = \text{opening cost of algorithm.}$$

The algorithm "touches" each facility location exactly once, either selecting or dropping it $\Rightarrow$ summing over all algorithm rounds and using Lemma 3.3 we get:

$$\text{Summed opening cost of algorithm} \leq \sum_{i \in \mathcal{F}} f_i \cdot y_i = F(x, y) \leq 2\, F(x^*, y^*) \tag{4}$$

Connection Cost: For all cities we either have

a) The city, j is assigned to a facility in its own neighbourhood: $\Rightarrow$ connection cost for $j \leq \Delta j$

b) The city, $\bar{j}$ is assigned to a facility in the neighbourhood of another city, j $\Rightarrow$ connection cost for $\bar{j} \leq \underbrace{\Delta\bar{j}}_{\text{to get to } B_j} + \underbrace{\Delta j}_{\text{to get to } j} + \underbrace{\Delta j}_{\text{to get to location } \phi(\bar{j})} \leq 3\Delta\bar{j}$ (see Figure 1 on the previous page)

Using a) and b) and observation 3.7 we get:

$$C(x'y') = \sum_{\substack{j \in \mathcal{C} \\ i \in \mathcal{F}}} c_{ij} \cdot x'_{ij} \leq \sum_{j \in \mathcal{C}} 3\Delta j \leq \sum_{j \in \mathcal{C}} 6\Delta j^* = 6 \cdot \sum_{\substack{j \in \mathcal{C} \\ i \in \mathcal{F}}} c_{ij} \cdot x^*_{ij} = 6\, C(x^*, y^*) \tag{5}$$

(4) and (5) gives: Algorithm cost $\leq 2\, F(x^*, y^*) + 6\, C(x^*, y^*) \leq 6\, Z(x^*, y^*) \leq 6\, OPT_{UMFL}$ $\square$