Nash equilibria in Voronoi Games on Graphs

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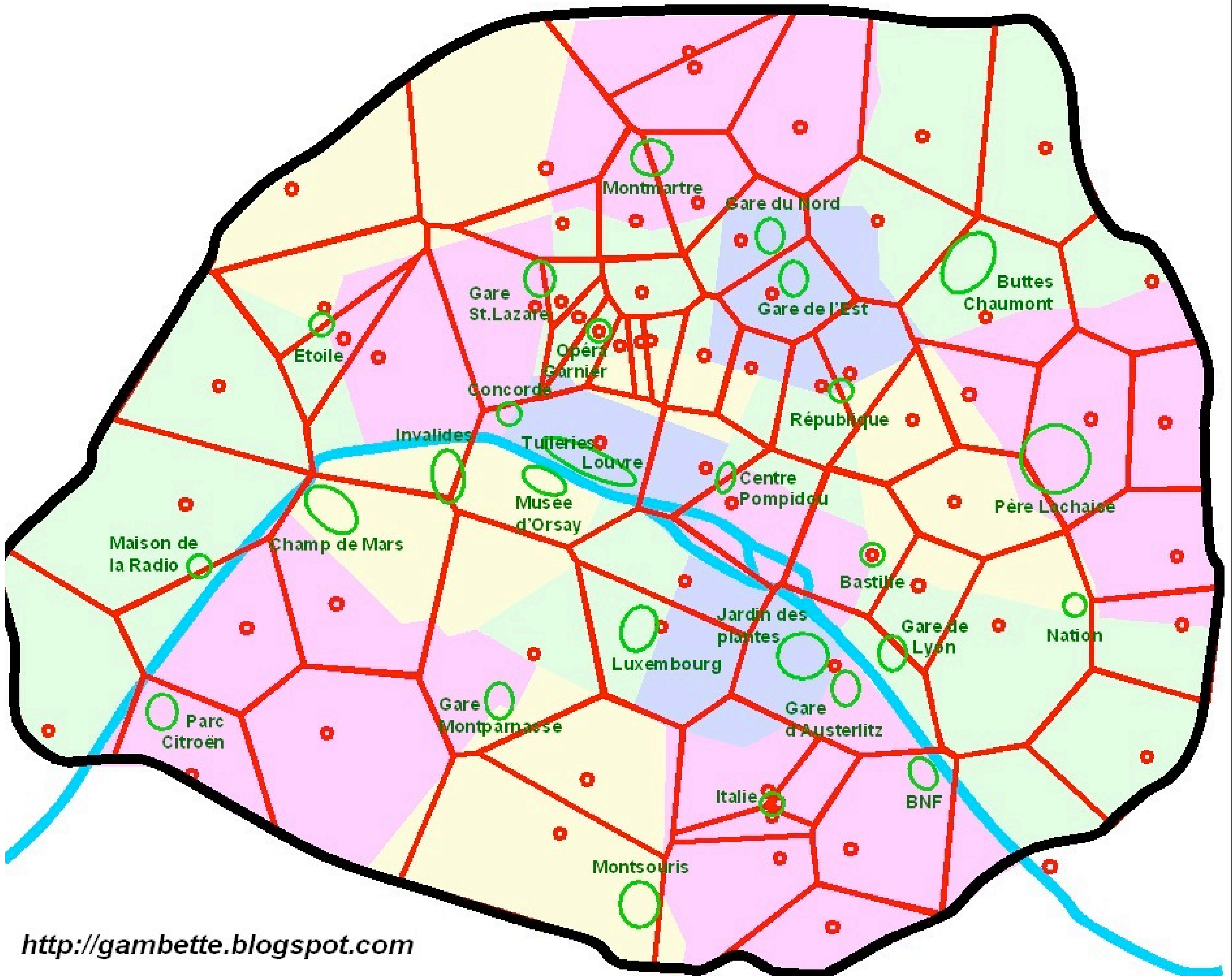
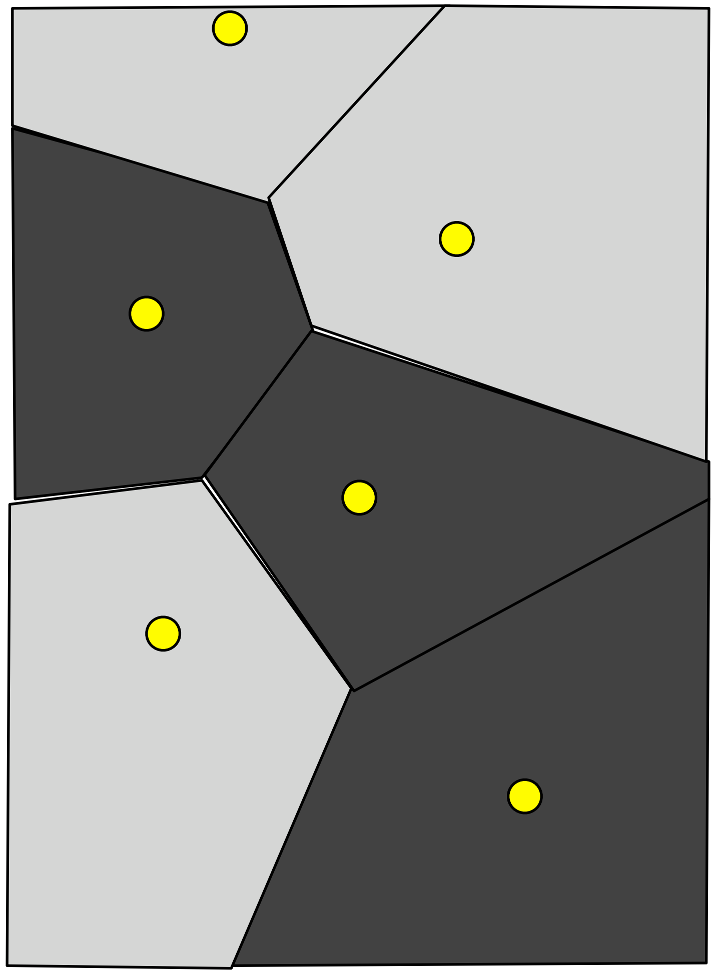
Plan

- *Motivation :* Study the interaction between selfish agents on Internet
- *k* players, each one chooses a vertex in graph *G* and gains the area of Voronoi cell.
- The existence of pure Nash equilibrium depends on *(k,G)* and deciding the existence is *NP*-complete.
- The difference of social cost between pure Nash equilibria is bounded by $\Omega(\sqrt{(n/k)})$, $O(\sqrt{(kn)})$

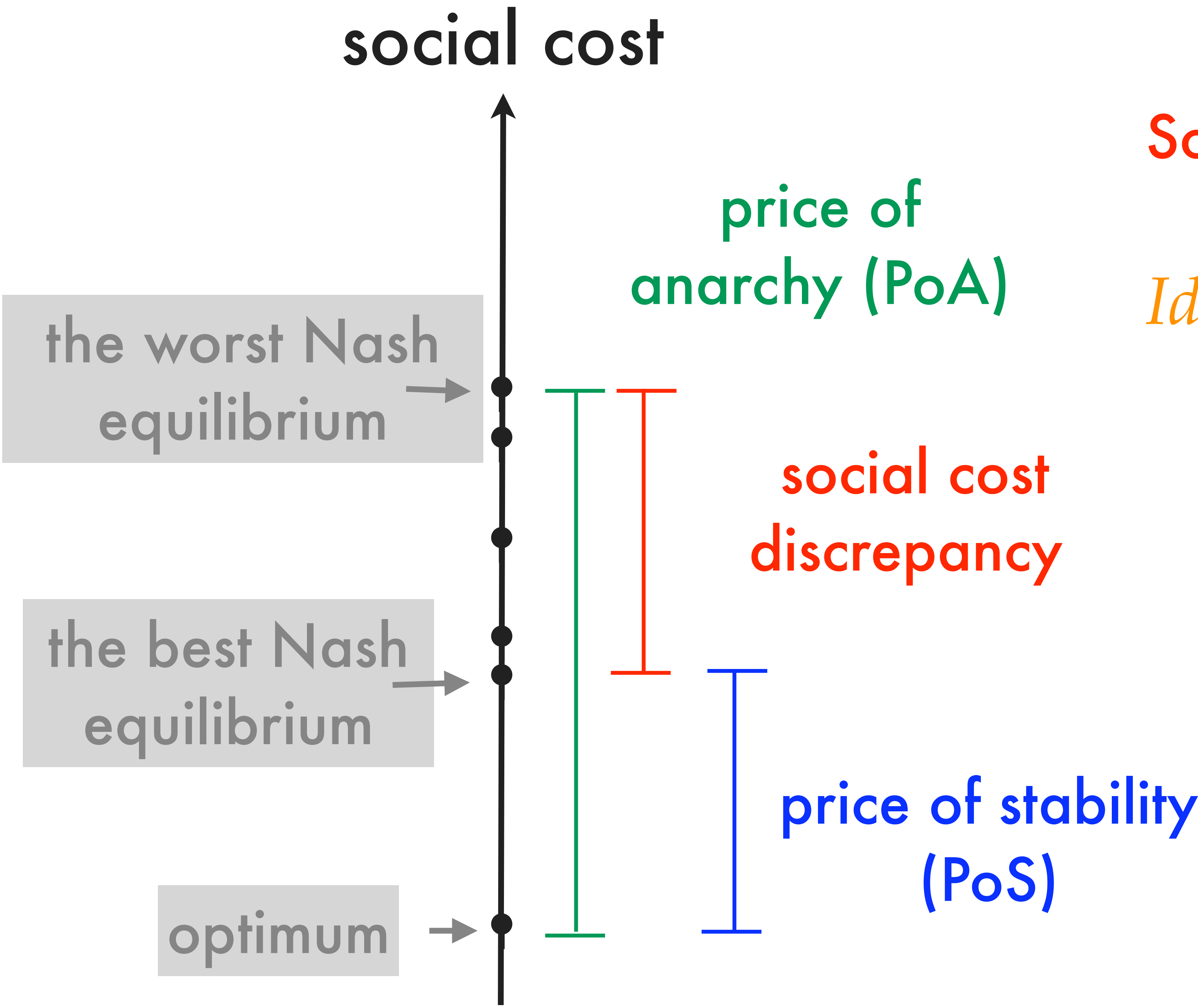


Related works

- Competitive facility location: Voronoi Games on continuous surface [Hee-Kap et al ’04, Cheong et al ‘04].
- Service Provider Games [Vetta’02]



Social cost discrepancy



Social cost discrepancy:

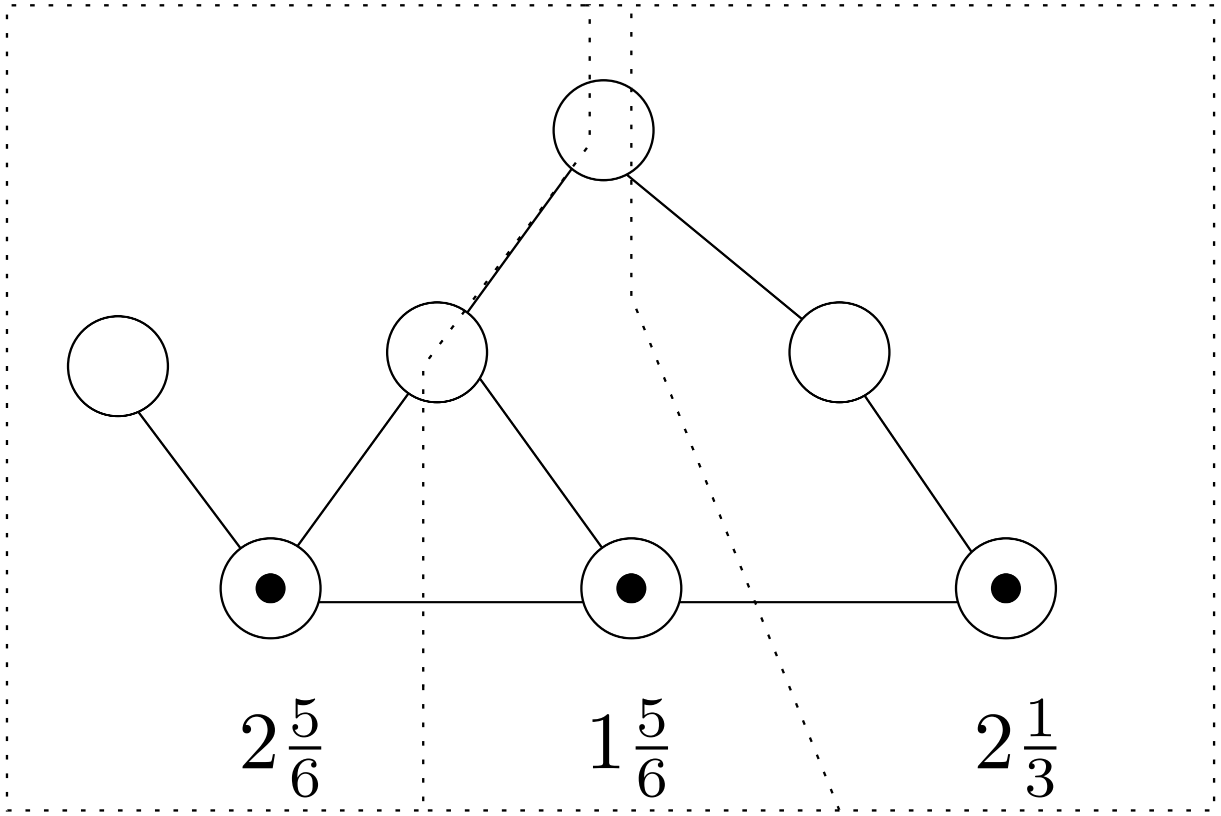
worst Nash / best Nash

Ideas:

- Unfair to compare the cost with OPT in selfish setting.
- Measure the degree of choice in a game.

The Game

- Given *G(V,E)*, *k* players. Player's strategy set is *V*.
- A vertex (customer) is assigned in equal fraction to the closest players.
- Payoff = the fractional amount of vertex assigned to the player.
- A pure Nash equilibrium is a strategy profile in which no one can unilaterally increase her payoff.



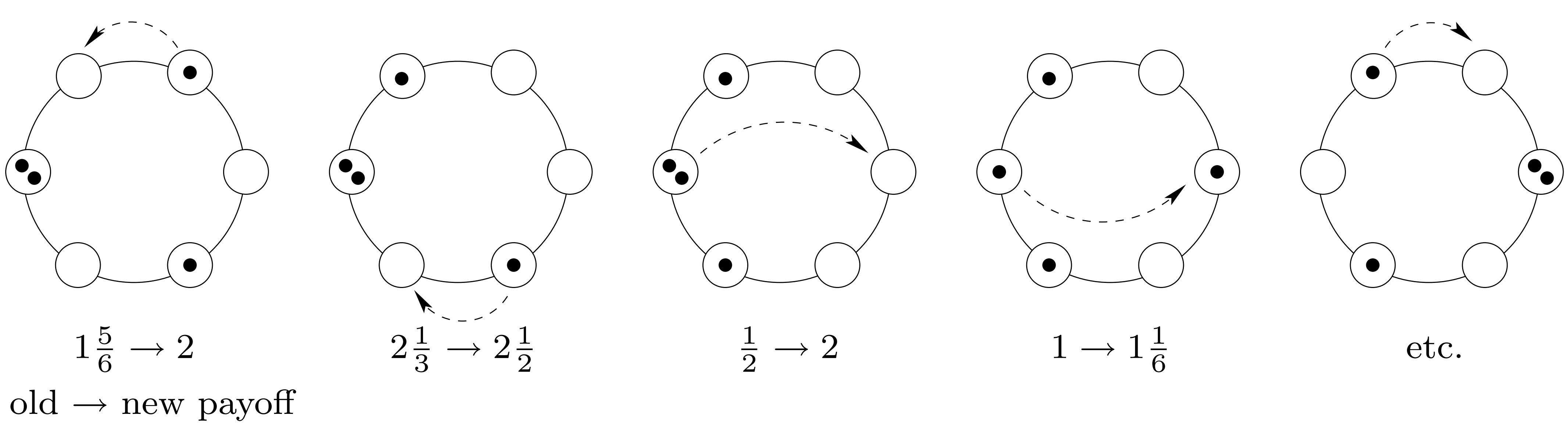
- Social cost = sum of distances over all vertices to their closest player = problem minimum k-median

Non convergence on the cycle

The game in continuous setting

- a player doesn't increase her payoff if she stays in the interval with the same neighbors,
- player A who moves to the same location as player B gains $1/2$ of the old gain of B.

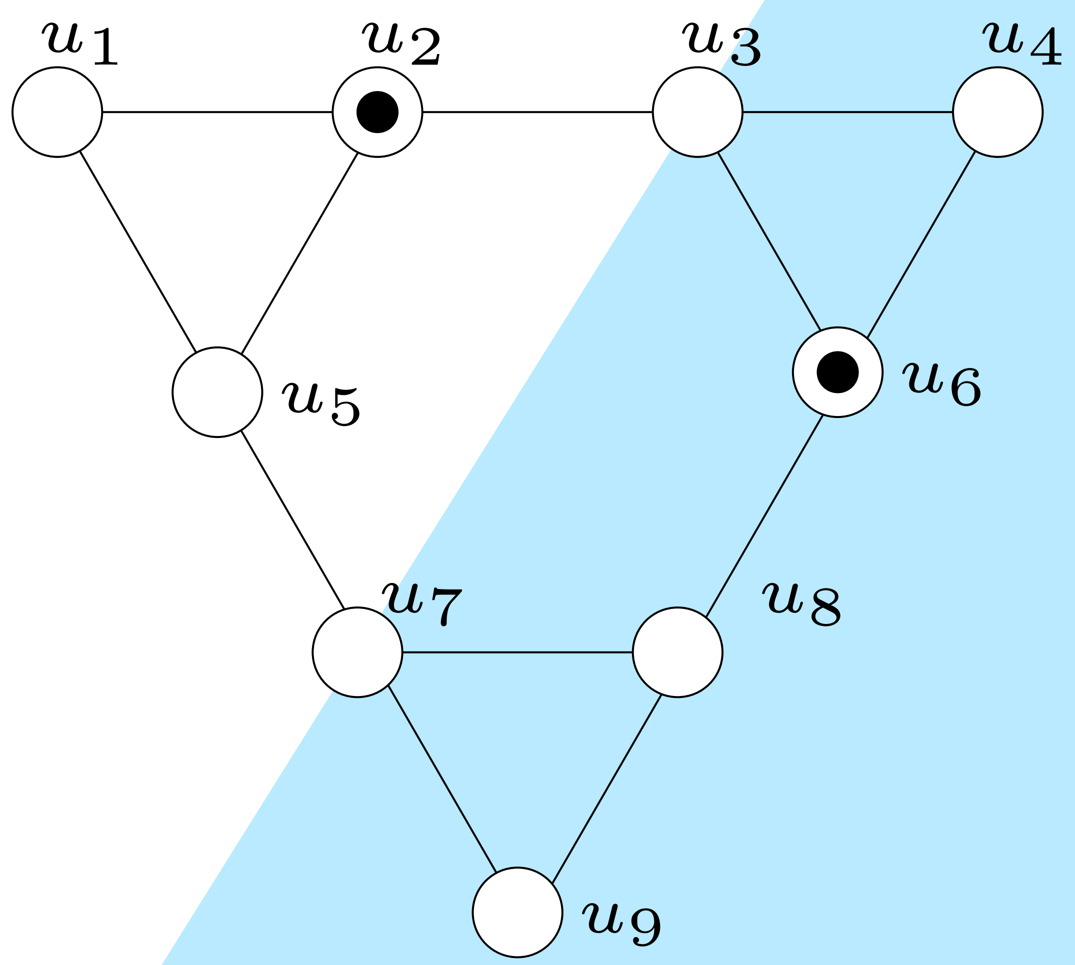
In discrete setting, it is different:



A gadget

Lemma: There is no Nash equilibrium with k=2 players.

Proof : By sym. player 1 is on u_2 (or u_1).
Then player 2 may gain 5 (or 6) by moving to u_6.
Now player 1 may increase his payoff by moving to u_7 and so on ...



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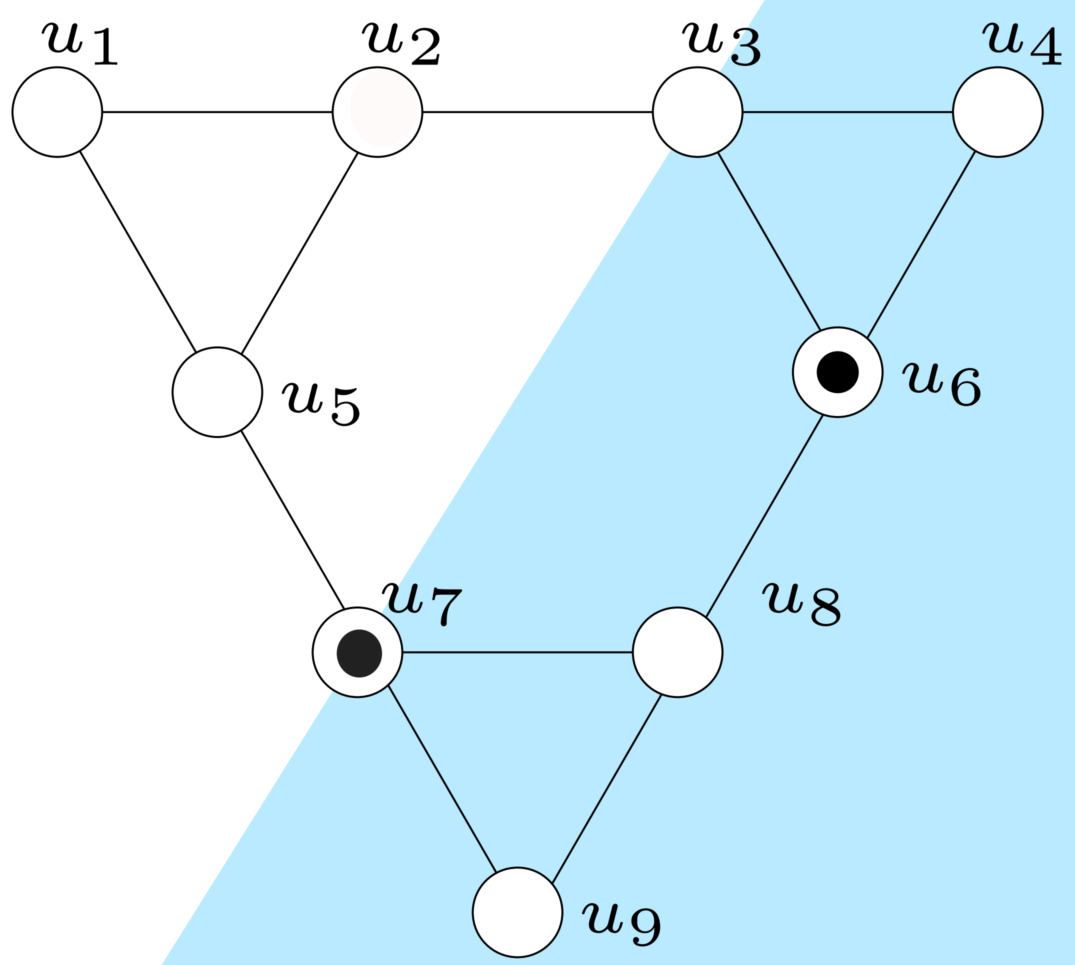

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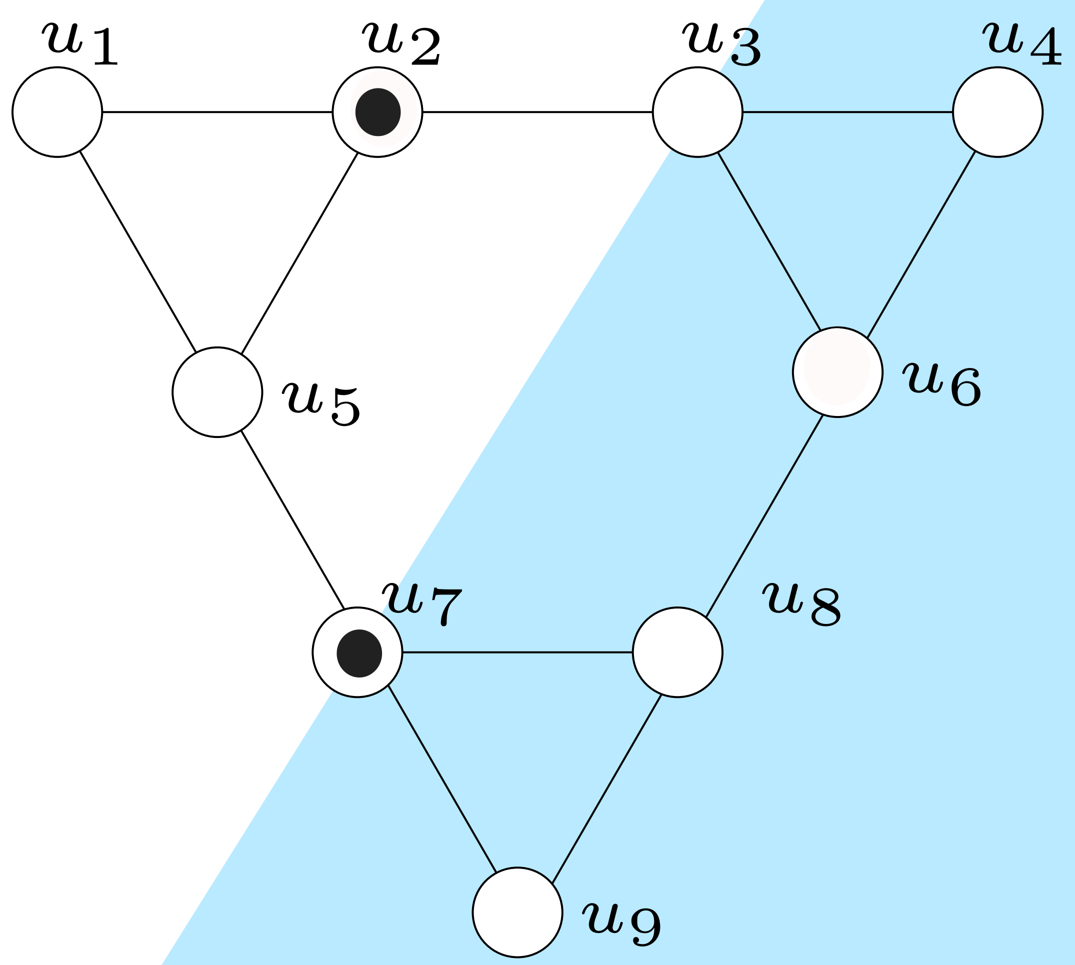


existence of equilibrium ?

Theorem:

Given $G(V,E)$ et k, deciding the existence of pure Nash equilibrium is *NP*-complete.

3-Partition (unary NP-hard)

input: $a_1,\ldots,a_{3n},B$ such that $\forall i\ B/4<a_i<B/2$, $\sum a_i=nB$

output: whether there exists a partition into *n* triplets, each of sum *B*

↓

General game (unary NP-hard)

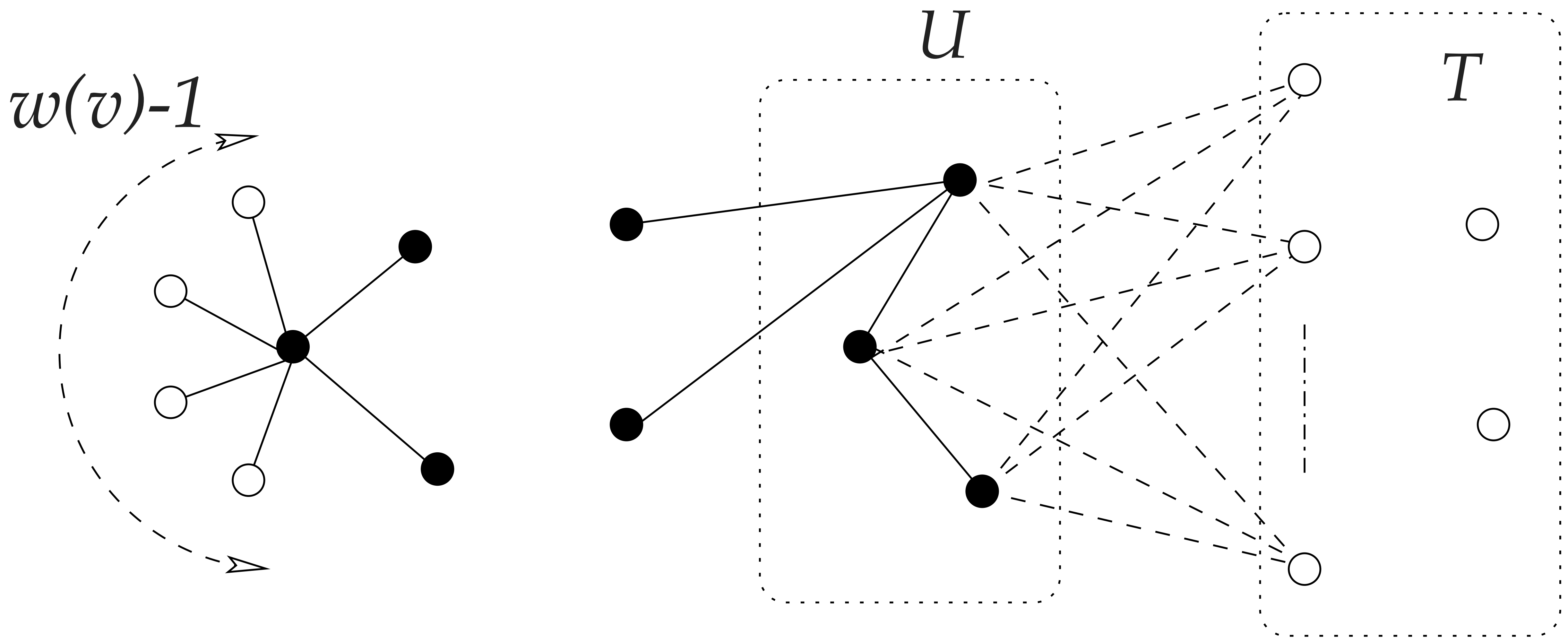
(positive weight *w* on vertices, strategy set is $U\subseteq V$)

↓

Original game (binary NP-hard)

General games

<*G(V,E),U,w,k*>: each vertex v has weight $w(v)$ and the strategy set is restricted to U.



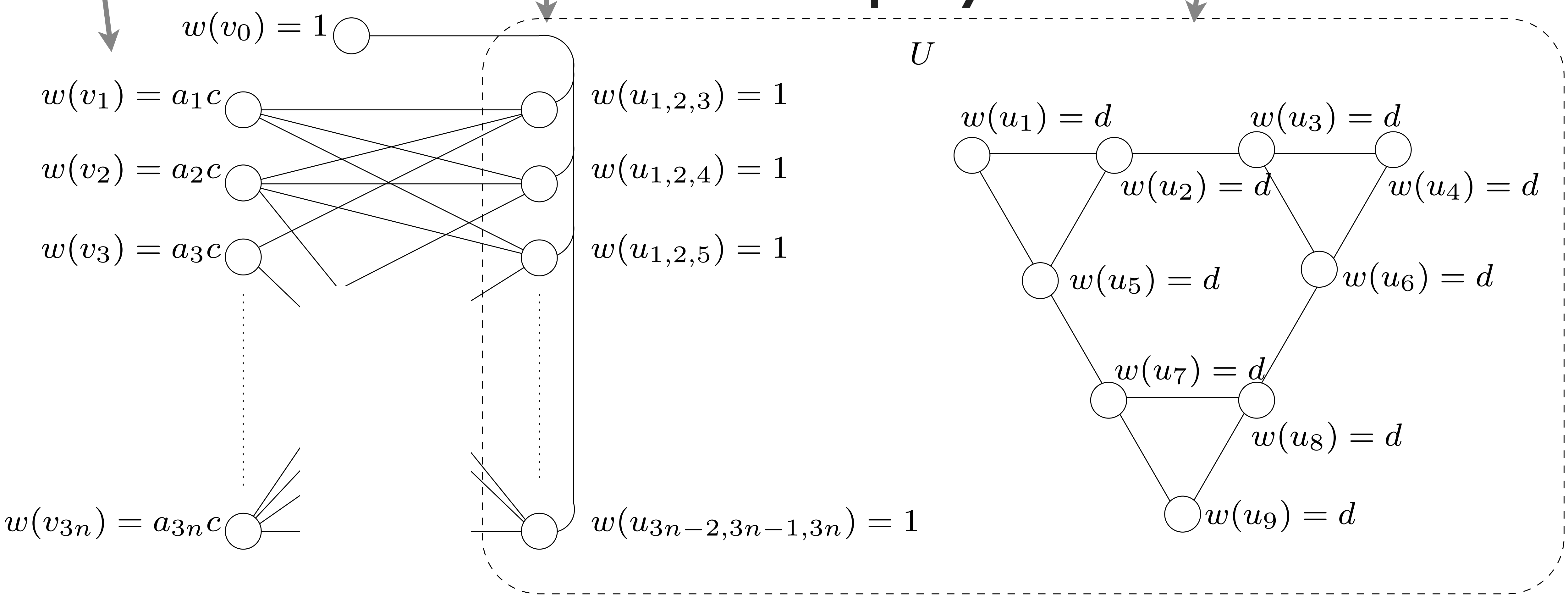
Proof Construction

vertices of weight $a_i c$

the gadget

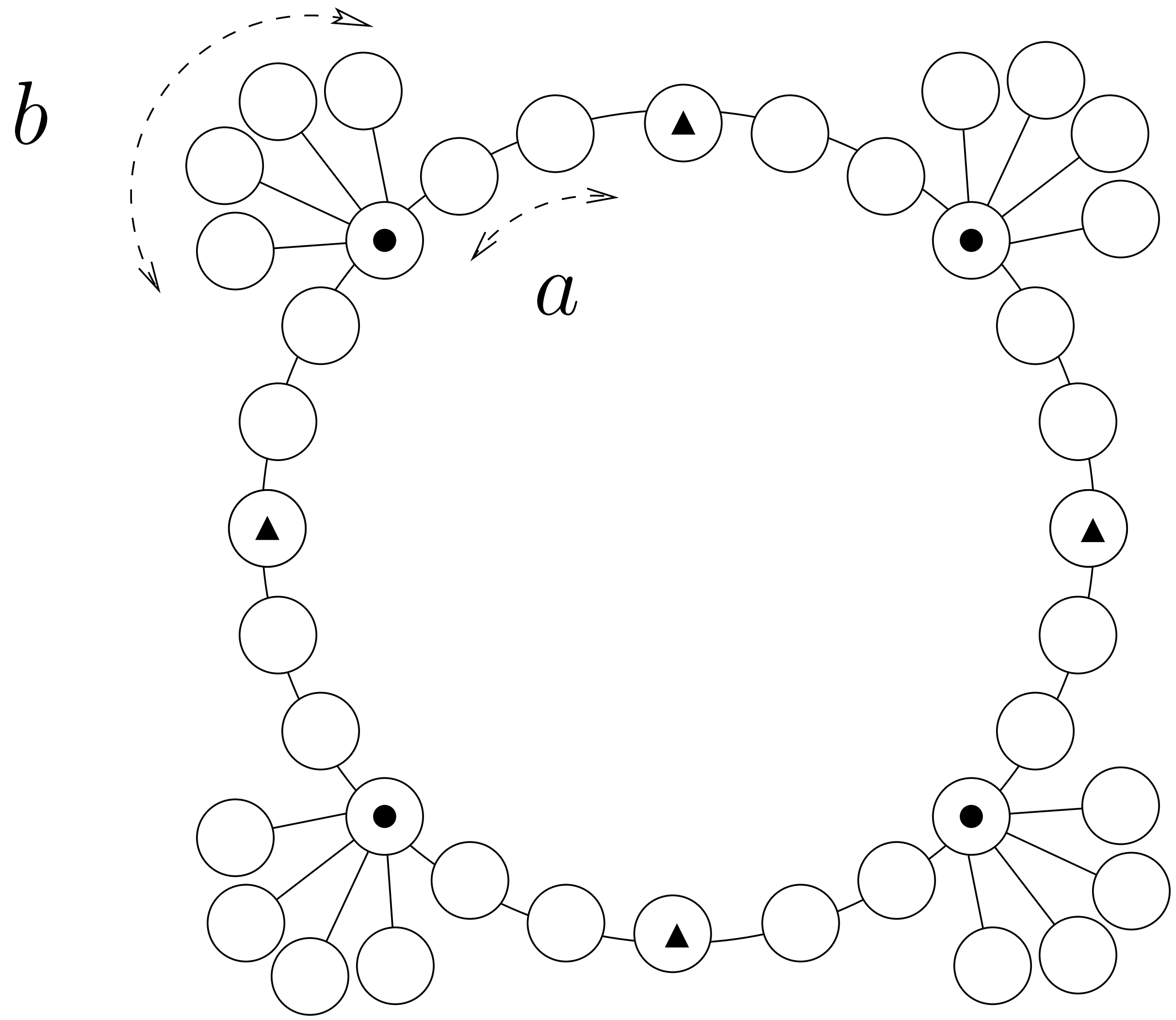
a vertex by triplet

$k=n+1$ players



where c,d are functions of n and B

Lower bound of Cost Discrepancy



equilibrium ● :

cost $\Theta(kb+ka^2)$

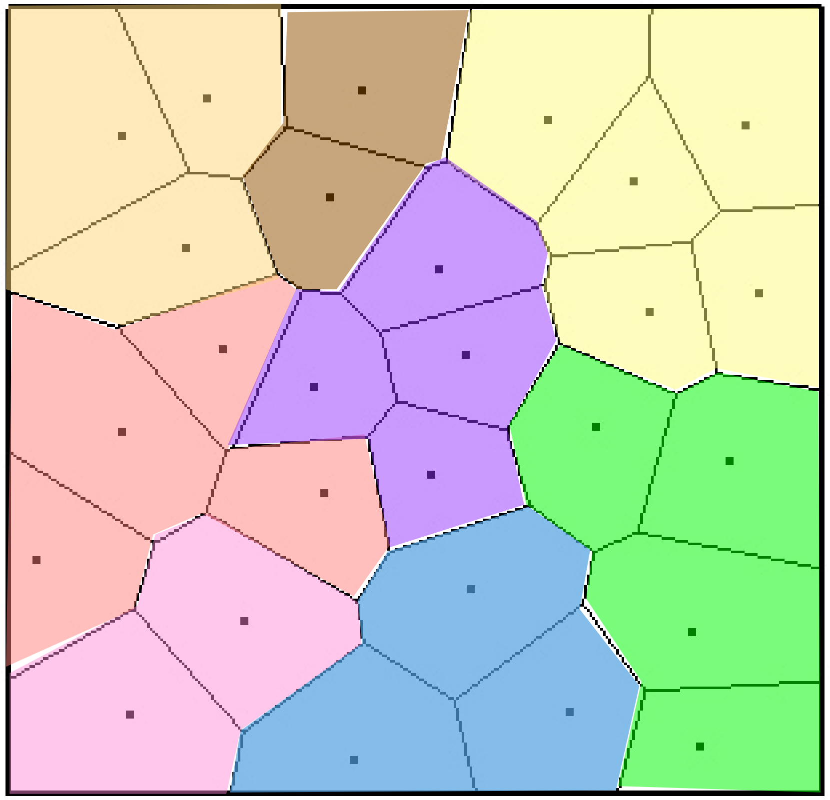
equilibrium ▲ :

cost $\Theta(kab+ka^2)$

worst ratio :

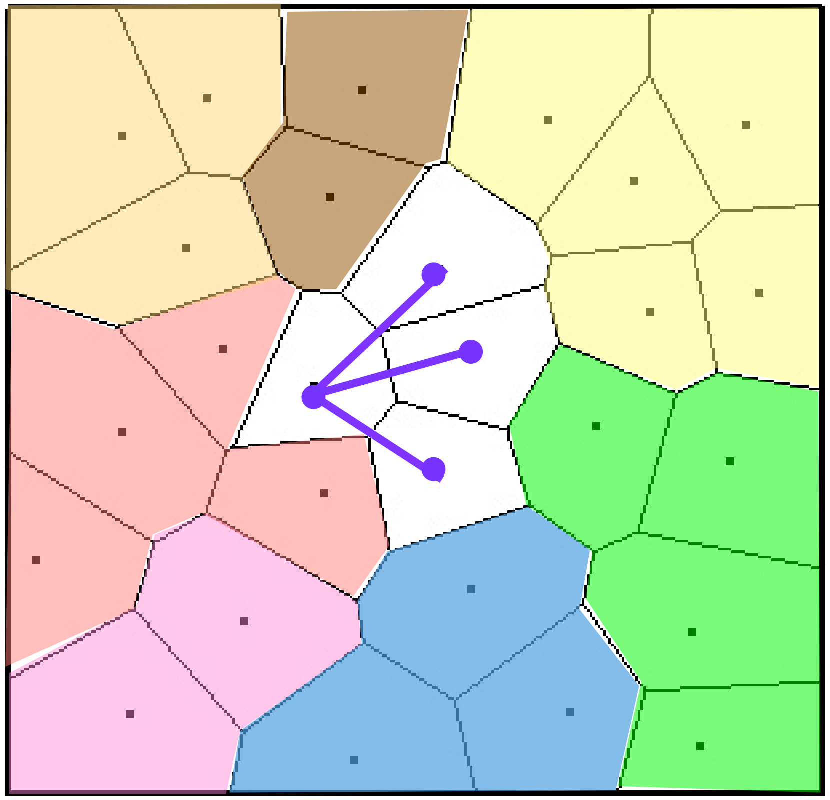
$\Omega(\sqrt{(n/k)})$ for $b=a^2$.

Upper bound of Cost Discrepancy



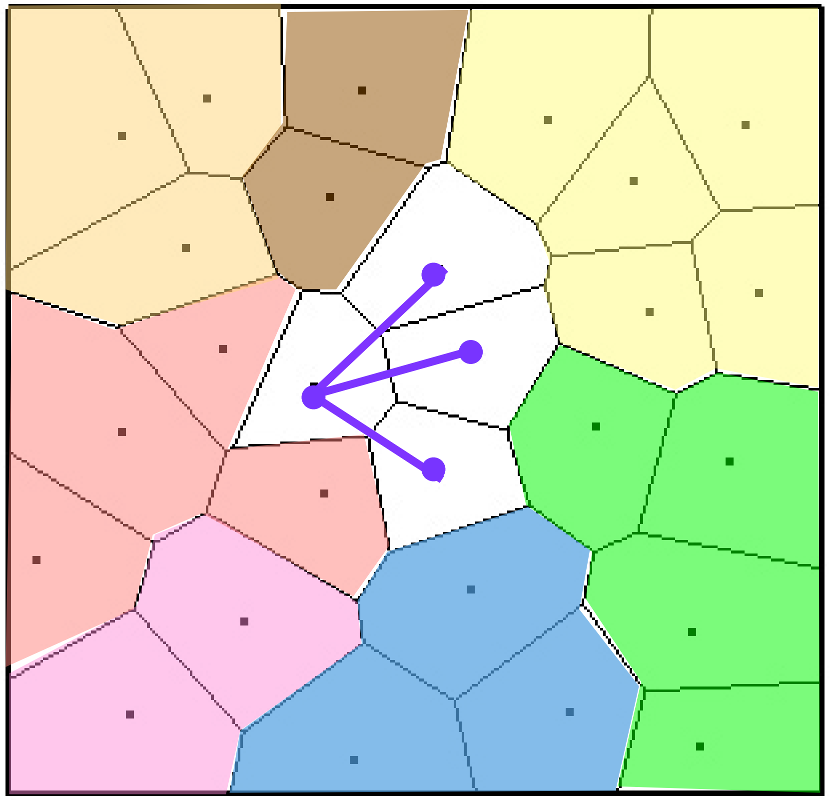
- Let ● and ▲ be two equilibria.
- *Idea*: these equilibria are not far from the one to the other.
- We group all Voronoi cell generated by ● into regions.

Delaunay graph – Stars



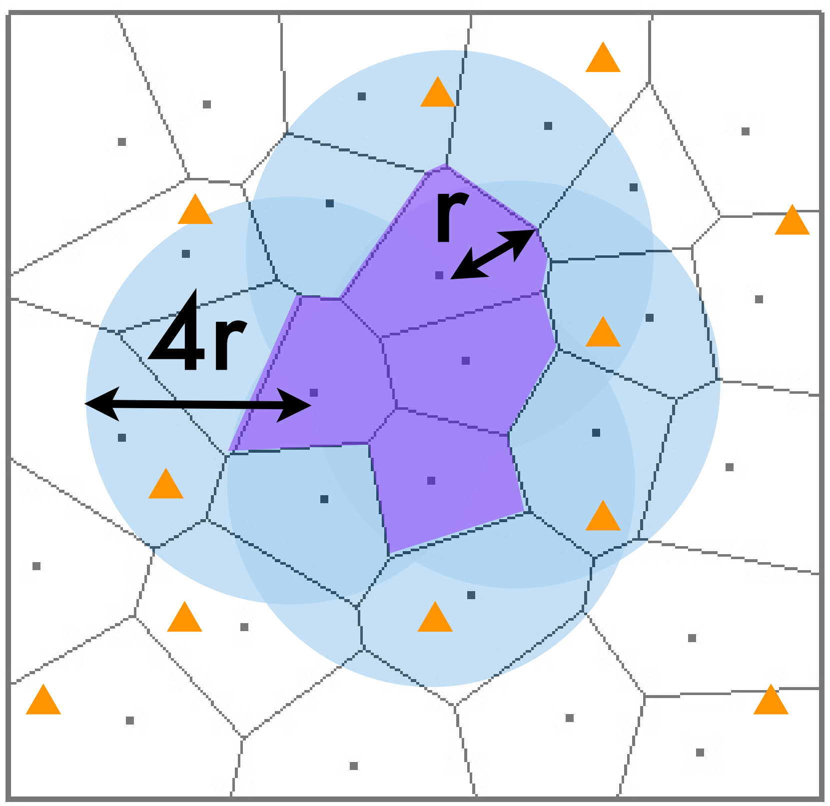
- *Delaunay graph:* $G(E,V)$ and an equilibrium •, (i,j) in H if they are neighbors.
- *Star:* $G(V,E)$, A is a star if $A \geq 2$ and ∃ a vertex in A connecting to all other vertices in A.

Delaunay graph – Stars



- *Delaunay graph:* $G(E,V)$ and an equilibrium •, (i,j) in *H* if they are neighbors.
- *Star:* $G(V,E)$, A is a star if $A \geq 2$ and ∃ a vertex in A connecting to all other vertices in A.
- *Fact:* Any connected graph can be partitioned into stars.

Upper bound of Cost Discrepancy



- For a fixed region, let r be the maximal distance of vertex-player.
- *Lemma:* there is at least one player of ▲ whose distance to a player ● of the star is at most $4r$.

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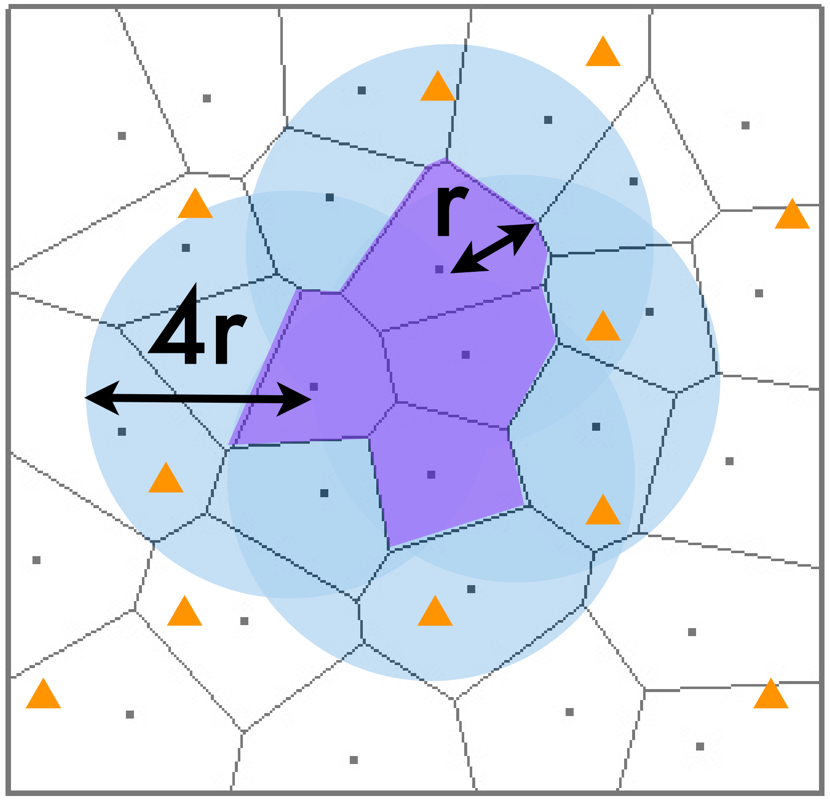

- For a fixed region, let *r* be the maximal distance of vertex-player.
- *Lemma:* there is at least one player of ▲ whose distance to a player ● of the star is at most *4r*.
- *Theorem:* for any connected graph *G(V,E)* with *k* players, the cost discrepancy is $O(\sqrt{(kn)})$.

And now...

- Close the gap between $\sqrt{(n/k)}$ and $\sqrt{(kn)}$
- Study the cost discrepancy in the others games.

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Thank you !