Tile Packing Tomography is NP-hard

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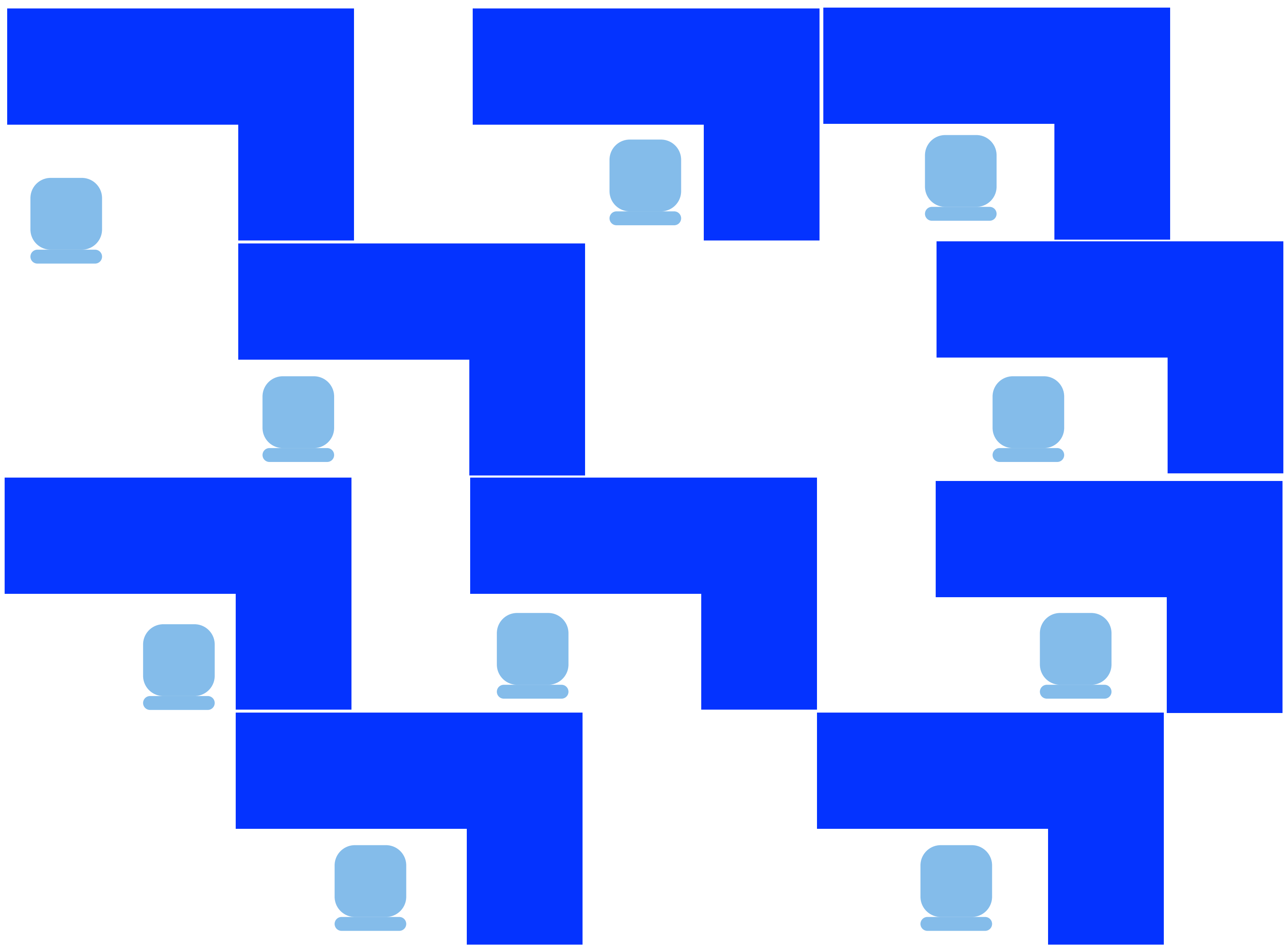
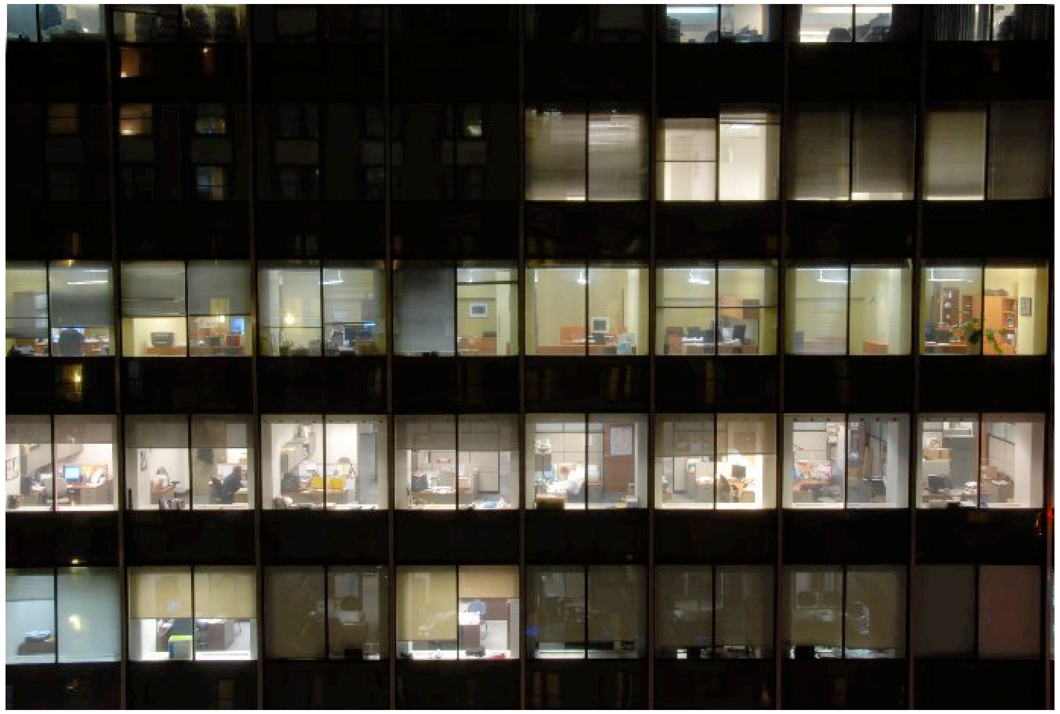
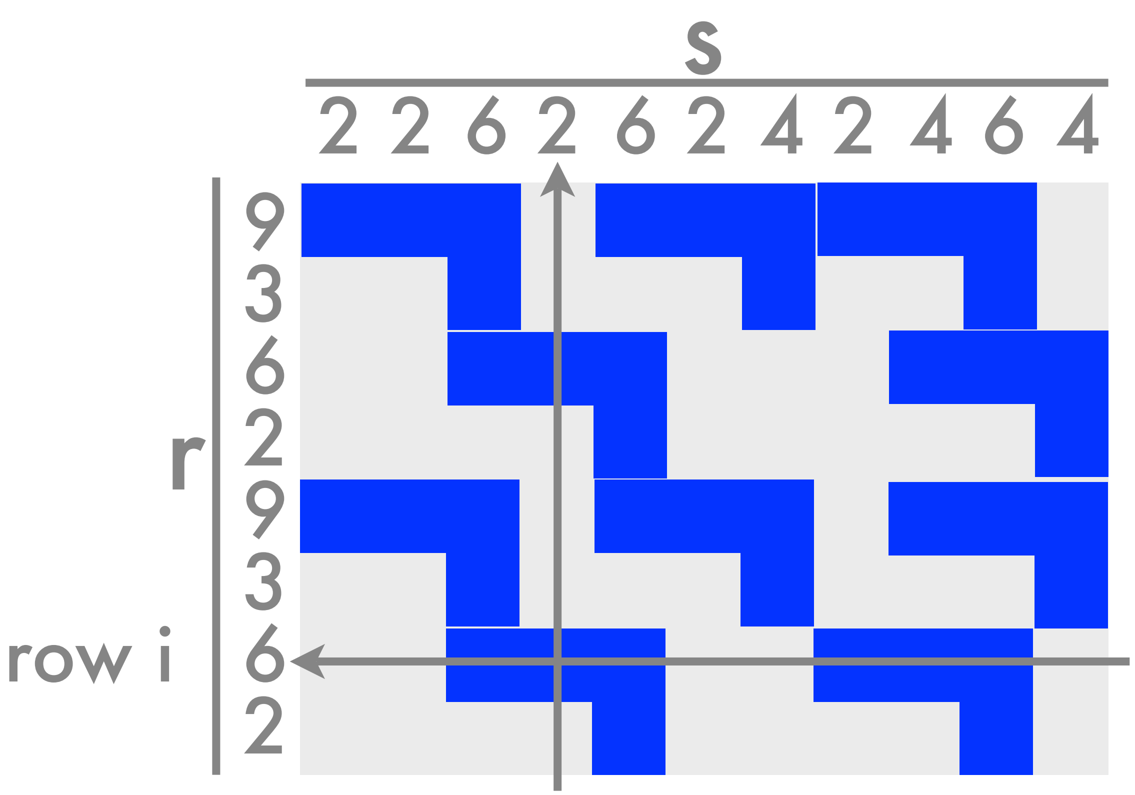
Tables in an open space office



- disjoint copies of the same shape
- same orientation

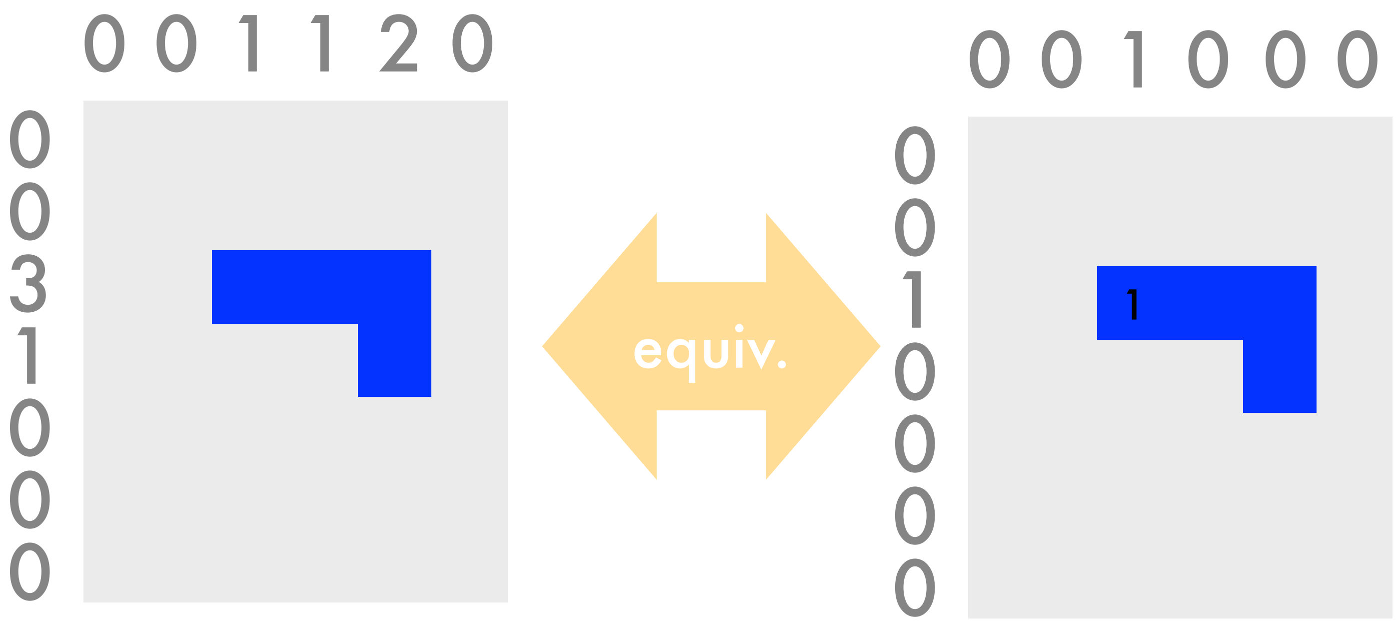
Non-intrusive measurement

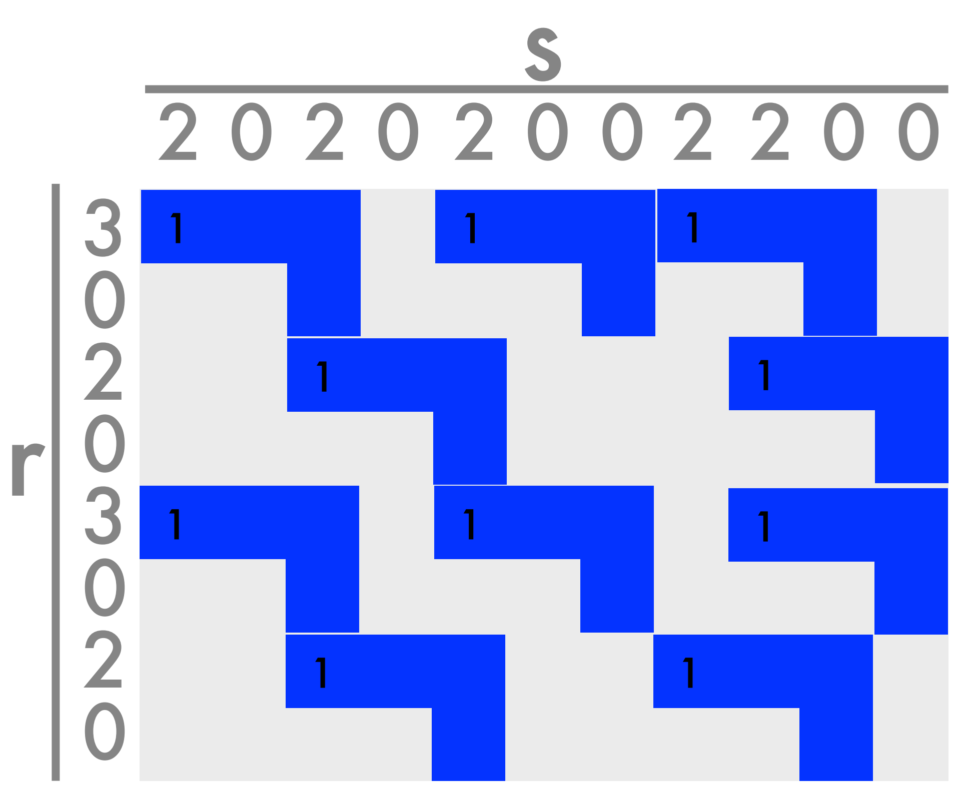




- the office is an n×m grid
- tables are aligned on the grid
- Measurement results in projection vectors r,s
- such that r_i is the number of grid cells of row i covered by a table (tile)
- same for columns

Equivalent measurement

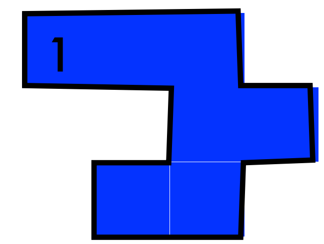




- alternative measurement (equivalent up to base change) :
- mark a cell in the tile
- projections count only marks

The tiling reconstruction pb

- A tile is a connected set of grid points



- Given a tile T, dimensions n,m and projections r, s
- does there exist a binary matrix M
- with $r_i=\sum_j Mij$, $s_j=\sum_i M_{ij}$
- and for $M_{ij}=1$, $M_{i'j'}=1$, the tiles T+(i,j) and T+(i',j') are disjoint ?

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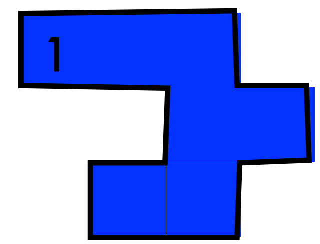

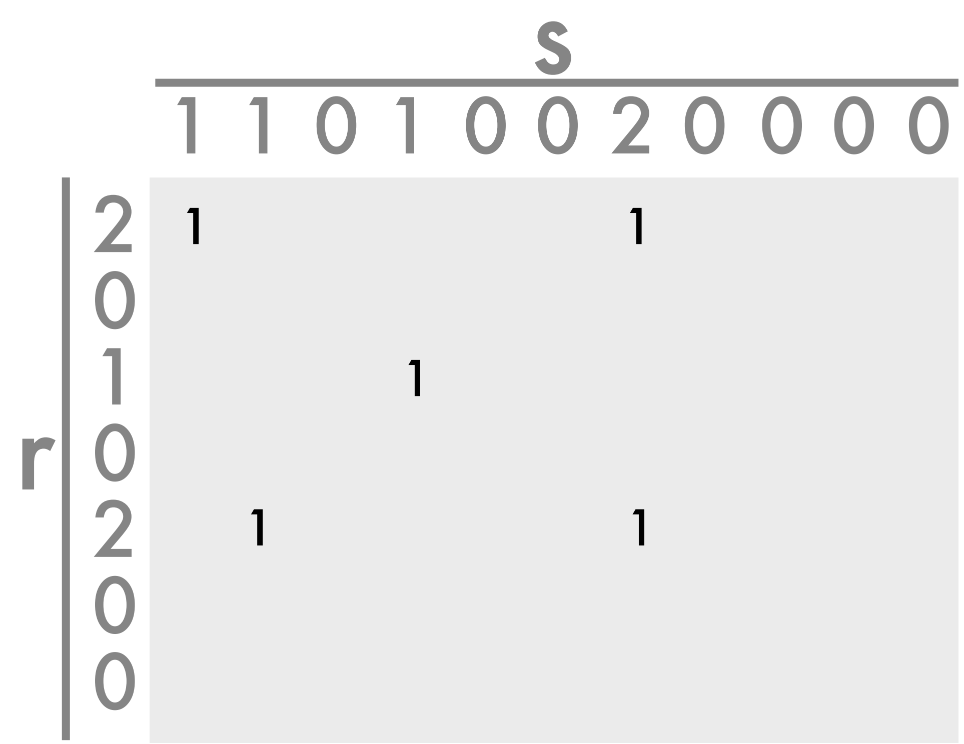

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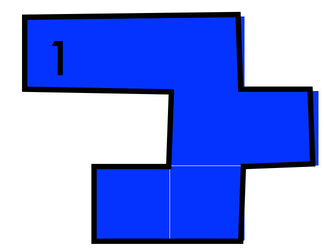
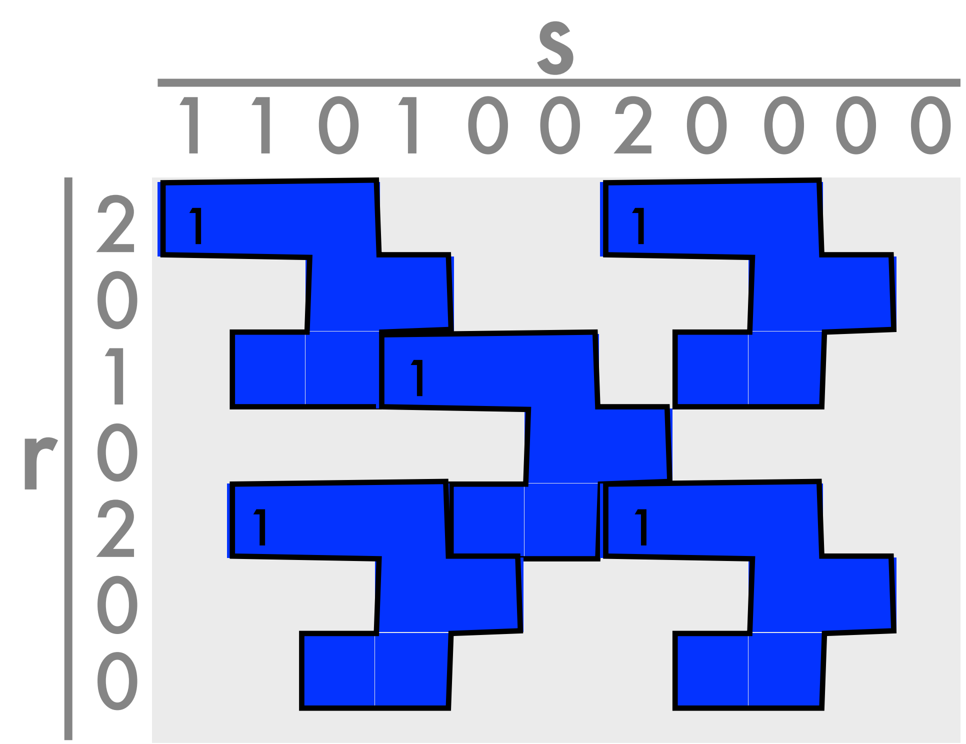




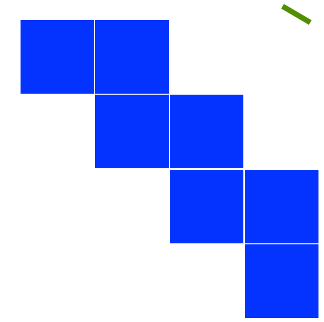
Complexity depends on T

- When the tile T is a bar, the problem is polynomial

- [this paper] When the tile T is not a bar, the problem is NP-hard

- [Ryser'63] Characterize r,c such that there is a binary matrix with projections r,c

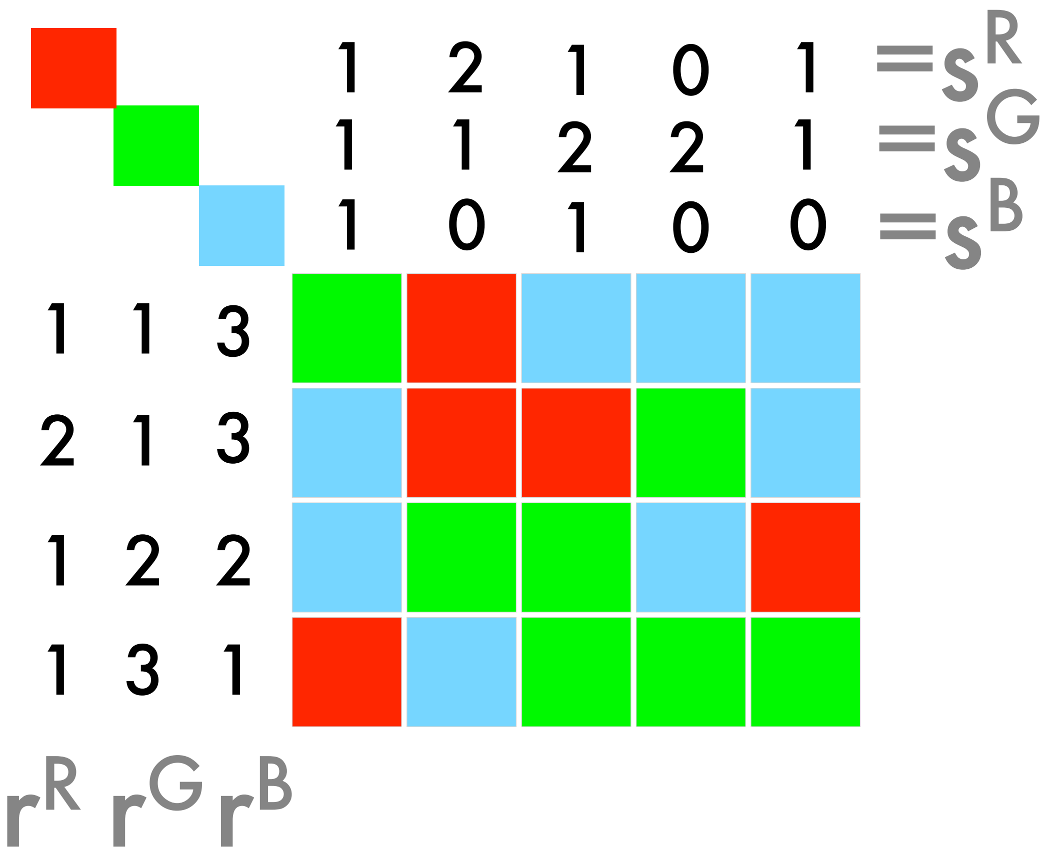
- [Picouleau'01] [D,Goles,Rapaport,Rémila'03] greedy algorithm to reconstruct tilings with bars



- [Chrobak,Couperous,D, Woeginger'03] NP-hardness for some very specific tiles

Related : 3-color tomography

- 3 colors {R,G,B}
- given projections r^c, s^c for every $c \in \{R,G,B\}$
- is there a matrix $M \in \{R,G,B\}^{n\times m}$
- such that $r^c_i = \#\{j : M_{ij} = c\}$ and $s^c_j = \#\{i : M_{ij} = c\}$ for every $c \in \{R,G,B\}$



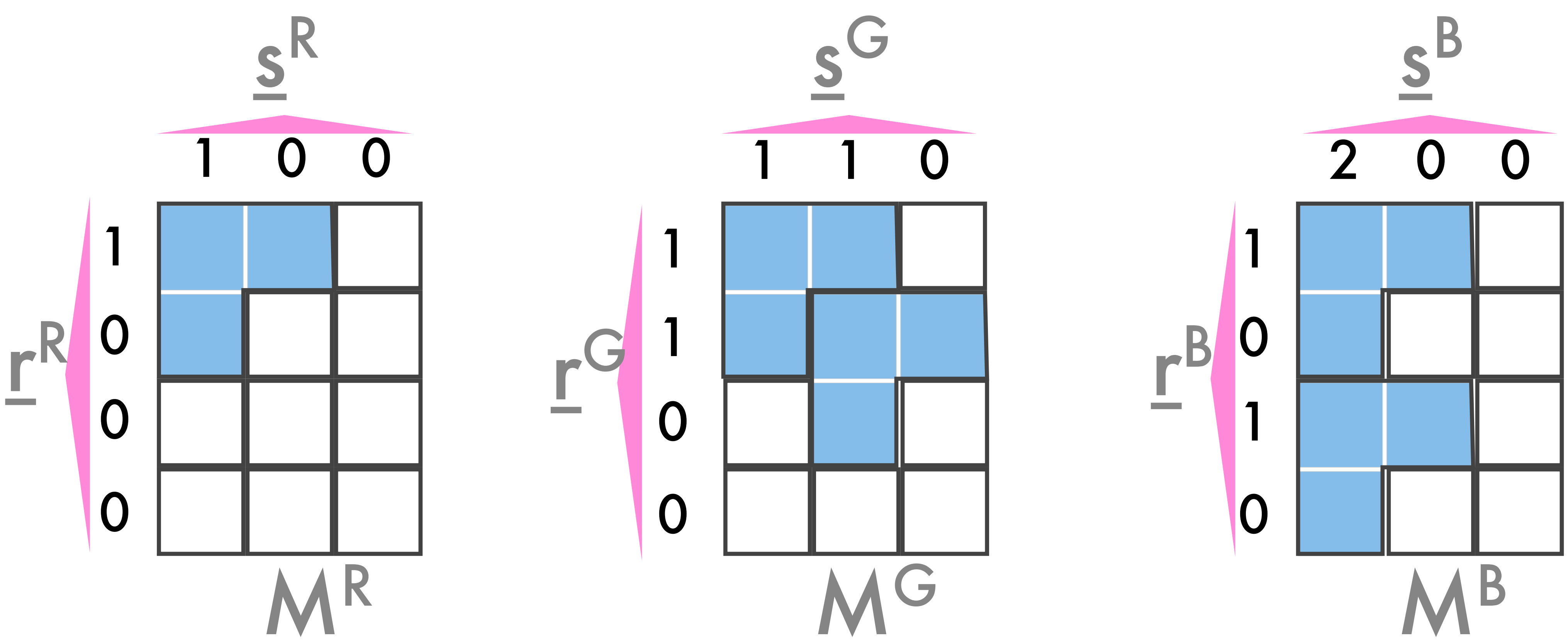
- [D,Guíñez,Matamala'09]
  3-color tomography is NP-hard

Reduction from 3-color tomography

- Reduce from 3-color tomography to tiling tomography

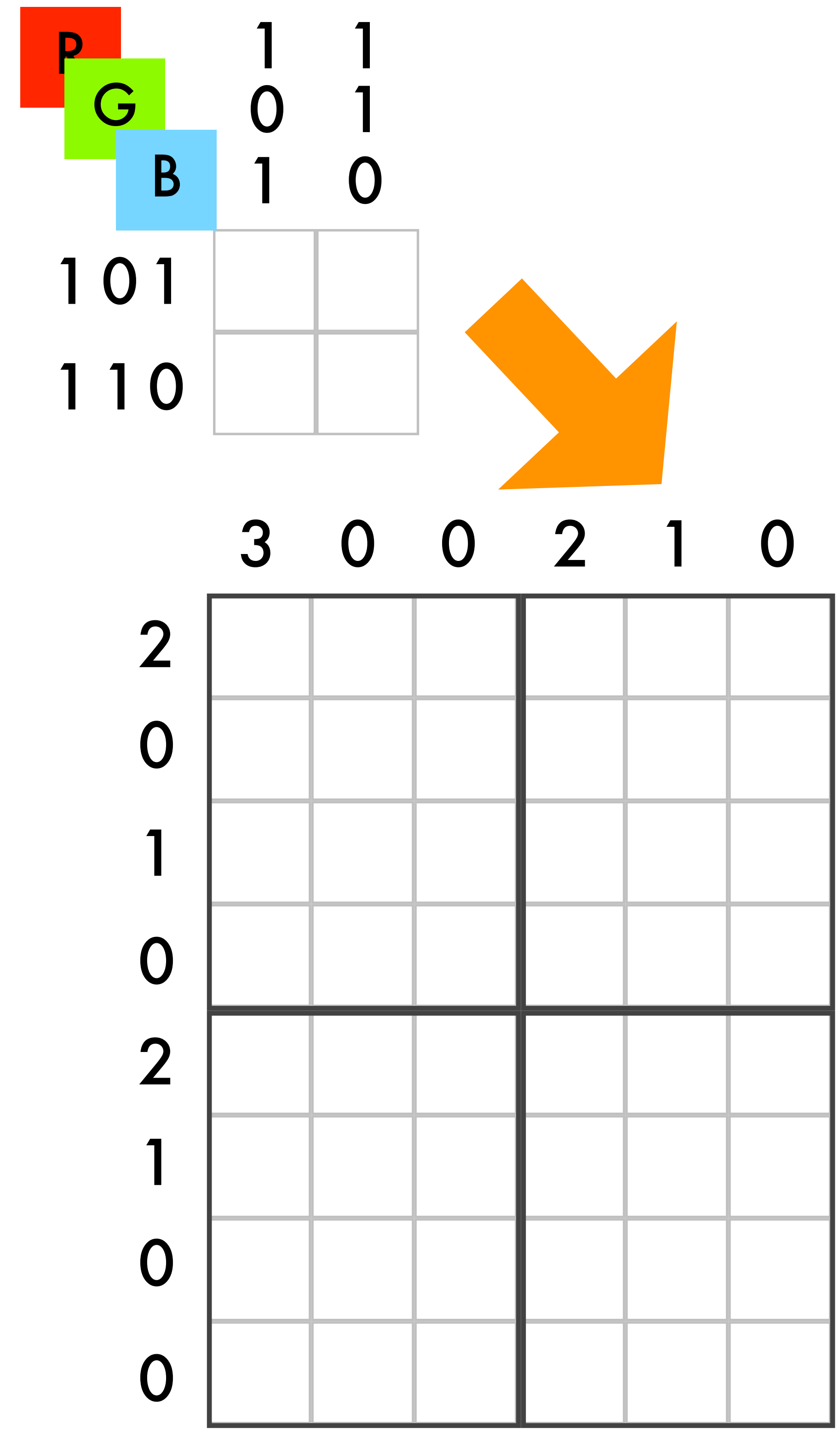
- Choose a block of fixed dimension k×l

- Choose 3 tilings of the block



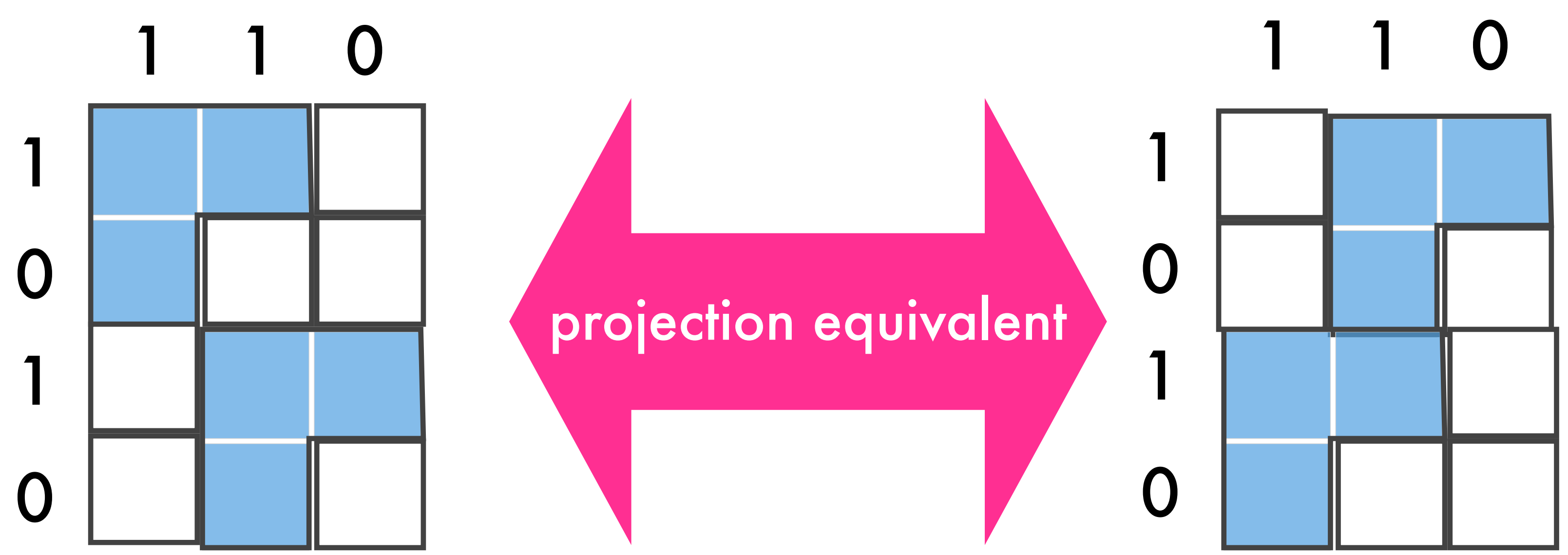
Reduction from 3-color tomography

- [in] instance r^c, s^c ($c \in \{R,G,B\}$) of the 3-color tomography problem for an n×m grid
- [out] instance r,s of the tiling tomography problem for an nk×ml grid such that projections of block row i are $r_i^R \cdot \underline{r}^R + r_i^G \cdot \underline{r}^G + r_i^Y \cdot \underline{r}^Y$ (same for columns)



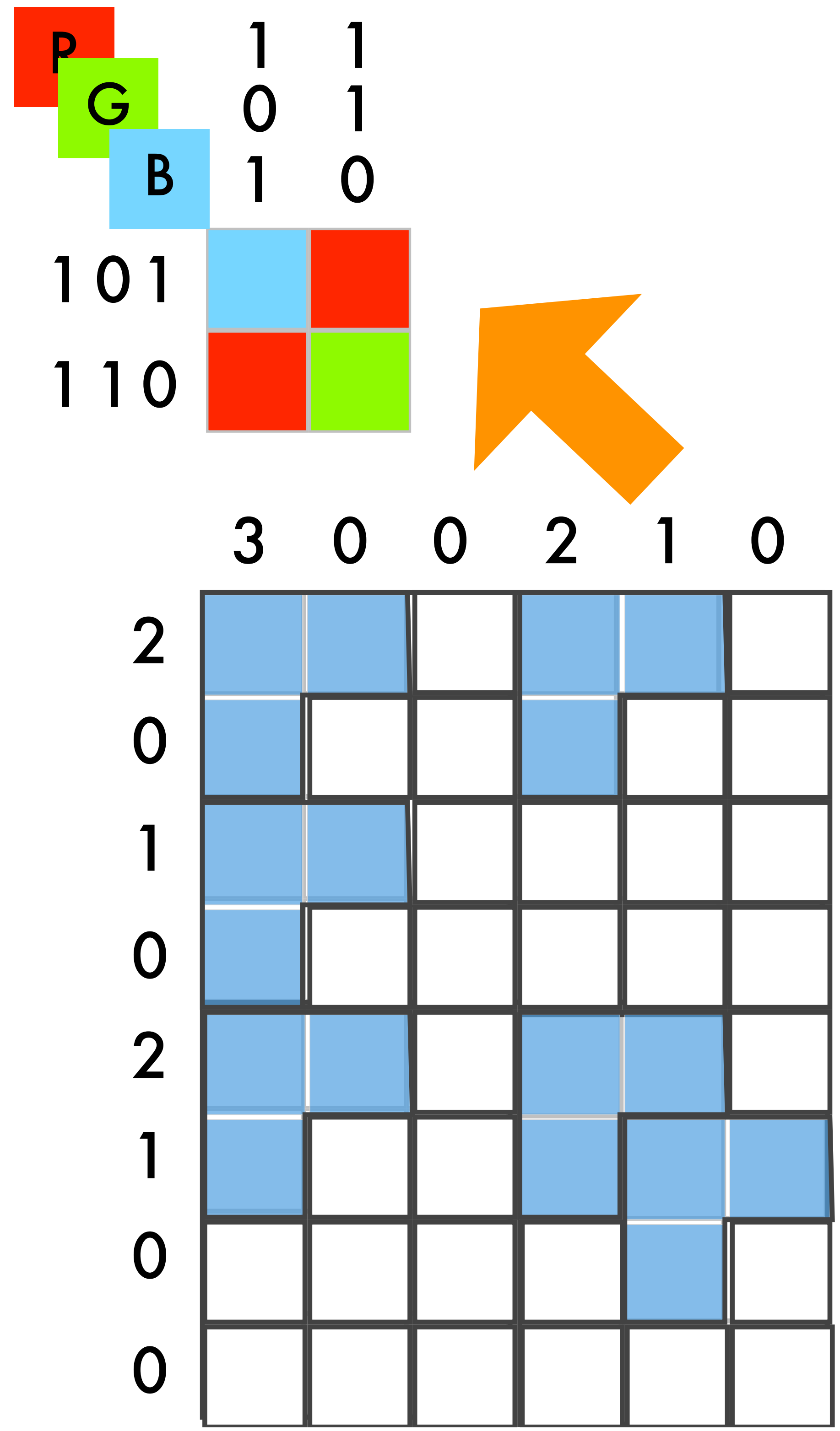
Requirements

- (R1) the row projections $\underline{r}^R, \underline{r}^G, \underline{r}^Y$ have to be affine linear independent
- (R2) Let M be a solution to the tiling tomography instance obtained by the reduction. Then every block in M is one of $\underline{M}^R, \underline{M}^G, \underline{M}^Y$ (or projection-equivalent)



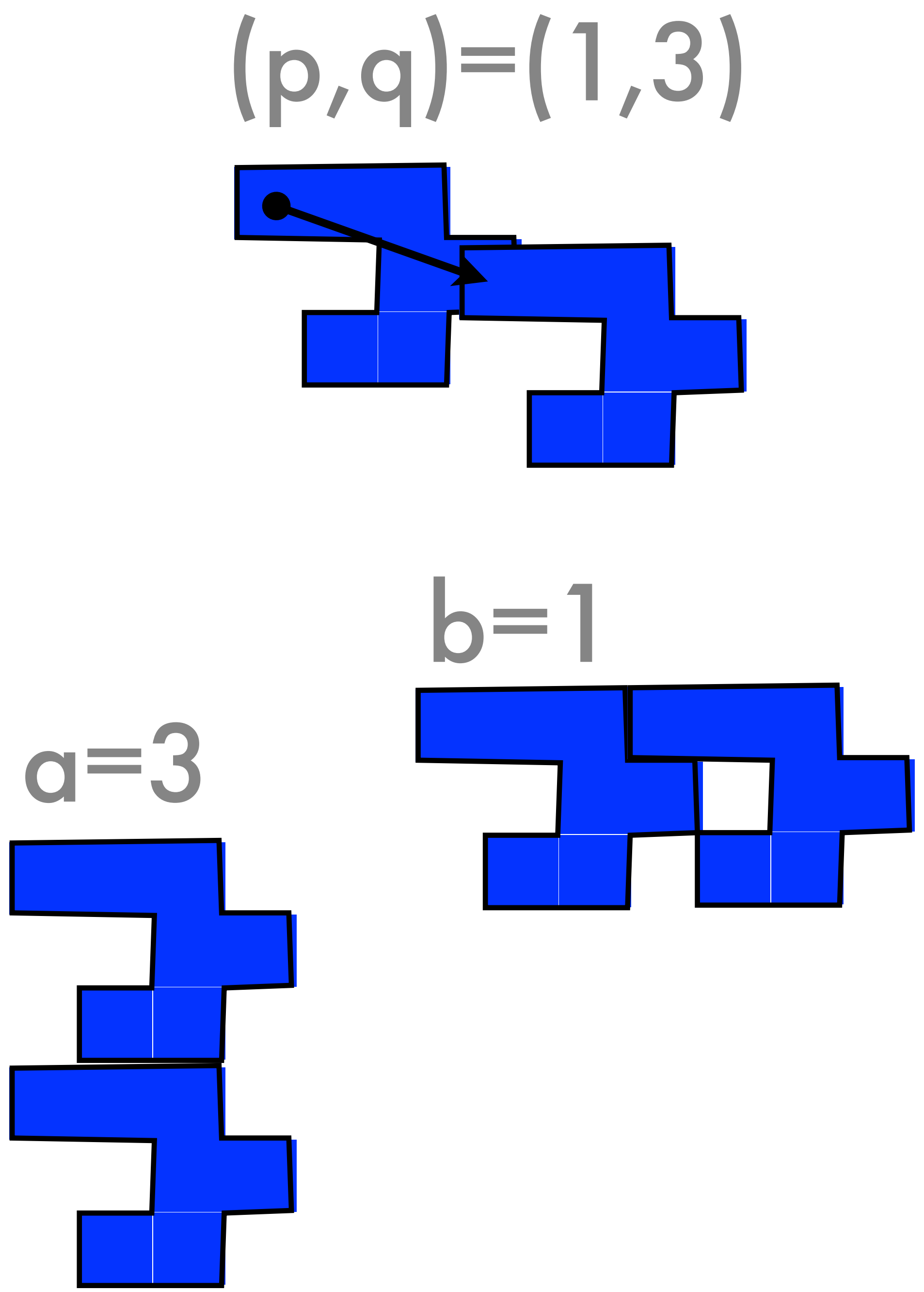
Implications

- (R2) ⇒ we can associate a color to every block in M
- and replace every block by a single colored cell (contract)
- (R1) ⇒ the obtained grid has the required projections, since any vector $n_R \cdot \underline{r}^R + n_G \cdot \underline{r}^G + n_Y \cdot \underline{r}^Y$ for $n_R + n_G + n_Y = n$ is uniquely decomposed into n_R, n_G, n_Y.



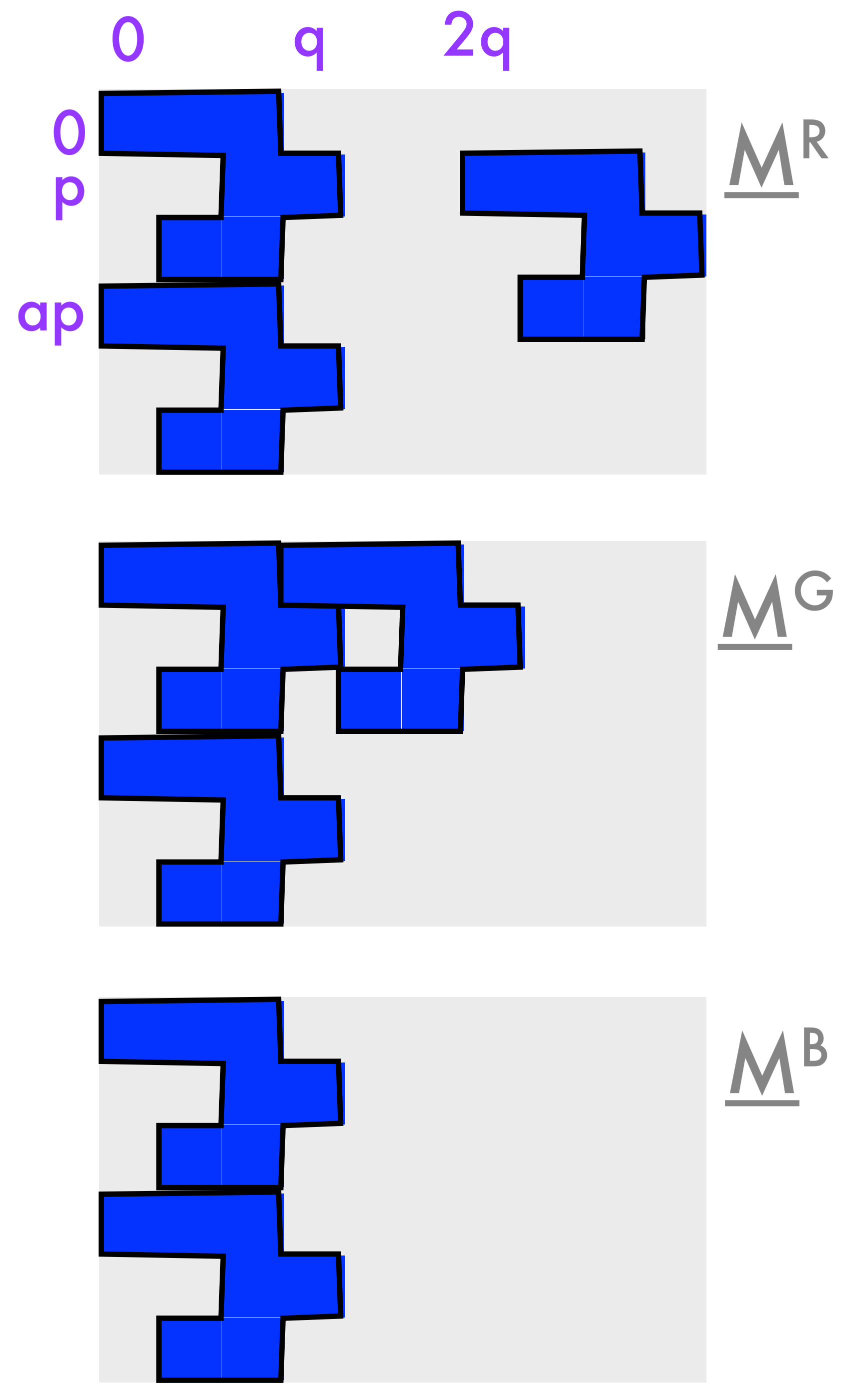
Apply this technique

- We divide the tiles into four classes
- and have a different construction for every class
- Fix a maximal conflicting vector (p,q)
- Choose smallest a>0 such that (ap,0) is not conflicting
- Choose smallest b>0 such that (0,bq) is not conflicting
- Cases are broken according to a,b,p,q



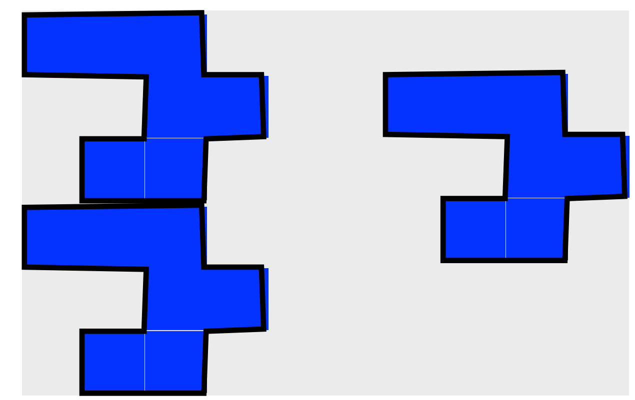
Example case b=1, a≥2

- We choose k,l large enough
- block tilings are as depicted, (R1) ok

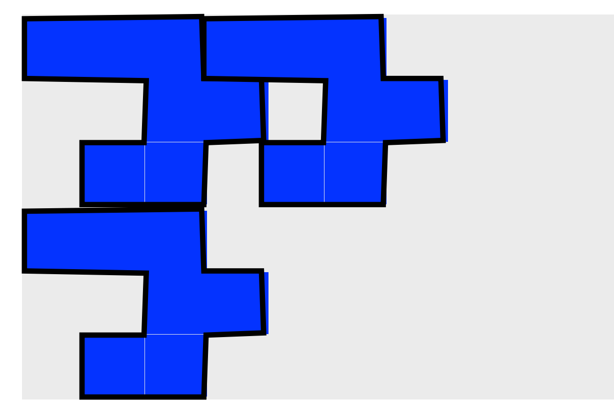


Proving (R2)

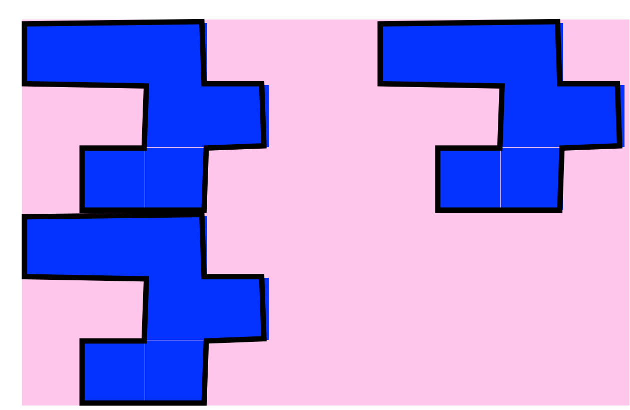
- We have to show
  (R2) Let M be a solution to the tiling tomography instance obtained by the reduction. Then every block in M is one of $\underline{M}^R$, $\underline{M}^G$, $\underline{M}^Y$ *(or projection-equivalent)*
- There might be another block tiling in the solution, namely $\underline{M}^A$
- It counts like $\underline{M}^R$ in the column projections and like $\underline{M}^G$ in the row projections
- Since total row projections equal total column projections this is impossible



$\underline{M}^R$



$\underline{M}^G$



$\underline{M}^A$

Perspectives

- What about approximation algorithms?
- What about complete tilings, for a constant number of tiles?