Pure equilibria: Existence and inefficiency & Online Auction

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Ecole Polytechnique
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Rational behaviors

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* What route to go to work?

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* Where to open a new competitive facility in Paris?

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* Where to open a new competitive facility in Paris?

* On Wednesday, what time to have lunch in Polytechnique?

Game Theory + Algorithms

* Entities in society, each with its own information and interests, behave in rational manners.

* Game theory is a deep theory studying such interactions (in economics, political science, ... etc).

* Theoretical computer science studies optimization problems, seeks to optimum, efficient computing, impossibility results, ... etc

Algorithmic Game Theory

* Research field on the interface of game theory and theoretical computer science (mostly algorithms)

* Formulating novel goals and problems, fresh looks on different issues (inspired by Internet, ...).

* The field has phenomenally exploded with many branches: computing Nash equilibrium, mechanism design, inefficiency of equilibria, ... etc

Motivation

* Pure equilibria: existence and inefficiency.

* Online Mechanism Design (Online Auction inspired by Google, Yahoo! Adwords, ...).

* Inspired by real problems.

* Mathematically beautiful.

Outline

- Voronoi Games on graphs
  - NP-complete whether there exists an equilibrium
  - Social cost discrepancy
- Scheduling Games in the Dark
  - Existence of equilibria
  - Optimal non-clairvoyant policy
- Online Algorithmic Mechanism Design
  - Truthful online auction with single-minded bidders

Voronoi Games on Graphs

Voronoi Games

- Summer holiday is also competition season.
- How to make this man happy?

Voronoi Games

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- Application: locations of supermarkets, Internet or mobile phone providers, ...

Voronoi Games

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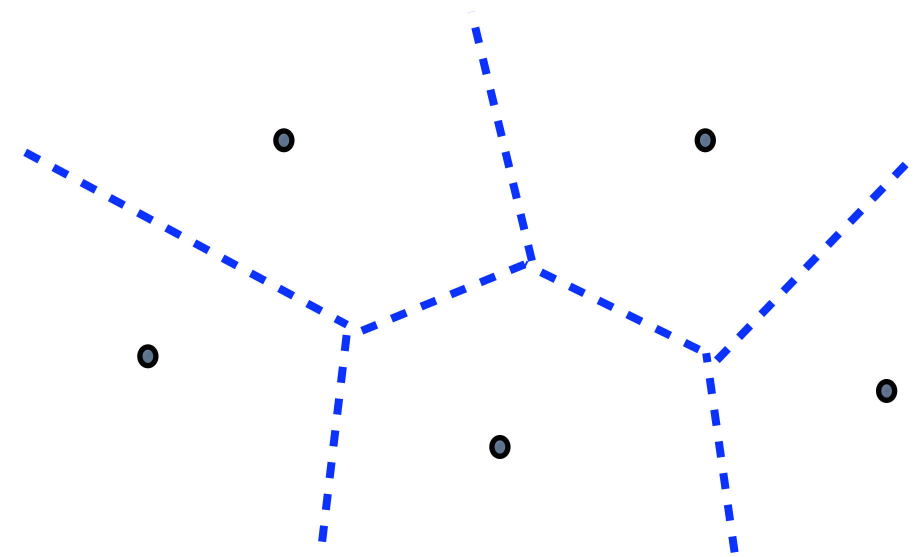


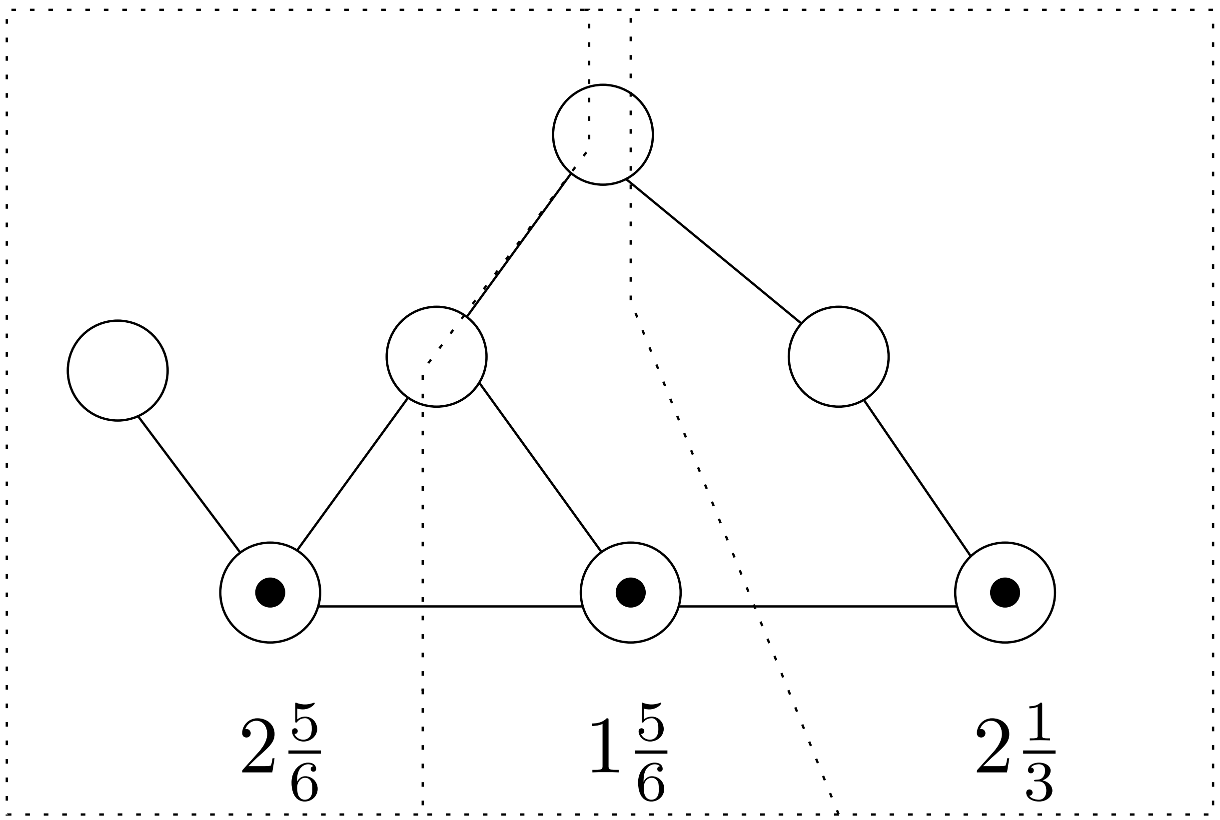
Voronoi Games on graphs

- Given $G(V, E), k$ players whose strategy set is V

- A vertex (client) is assigned in equal fraction to the closest players

- Utility = fractional amount of vertices assigned to the player.

- Social cost = sum of distances over all vertices to the closest player. (k-median optimization problem)



Equilibrium and Complexity

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Mixed equilibrium

Pure equilibrium

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choose a distribution over strategies

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deterministically choose a strategy

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always exists (by Nash)

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Existence: NP-hardness

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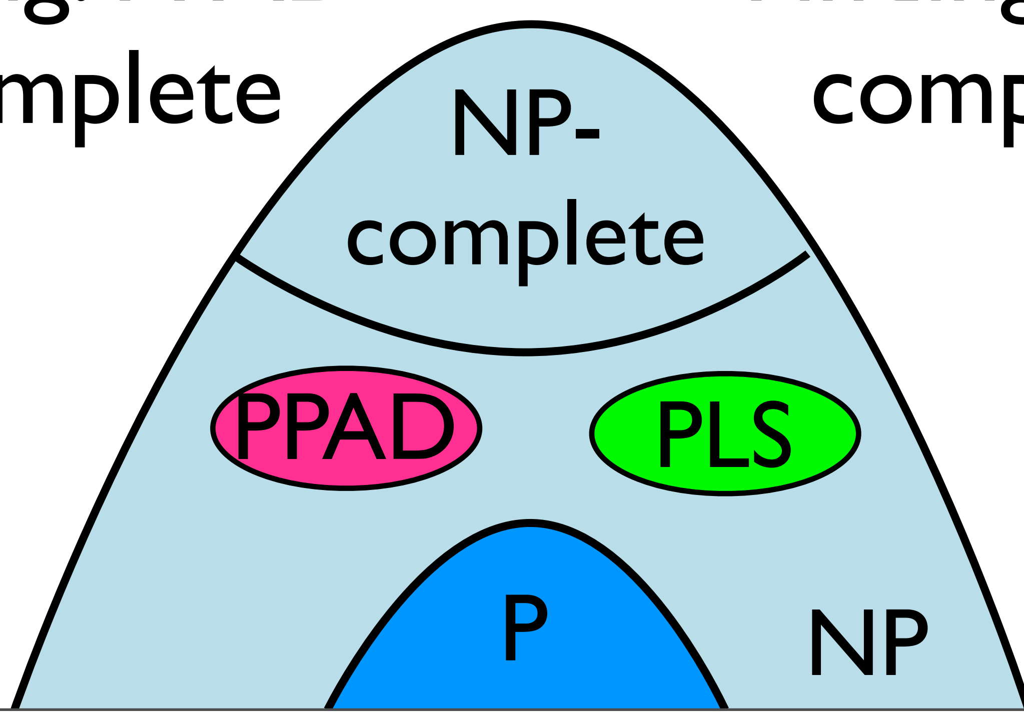

Framework in proving NP-hardness

Negated gadget for property *P* of a game + A larger game which encodes a NP-hard problem

→ NP-hardness in deciding whether a game possesses property *P*

Framework in proving NP-hardness

"counter example"

Negated gadget for property *P* of a game

+

A larger game which encodes a NP-hard problem

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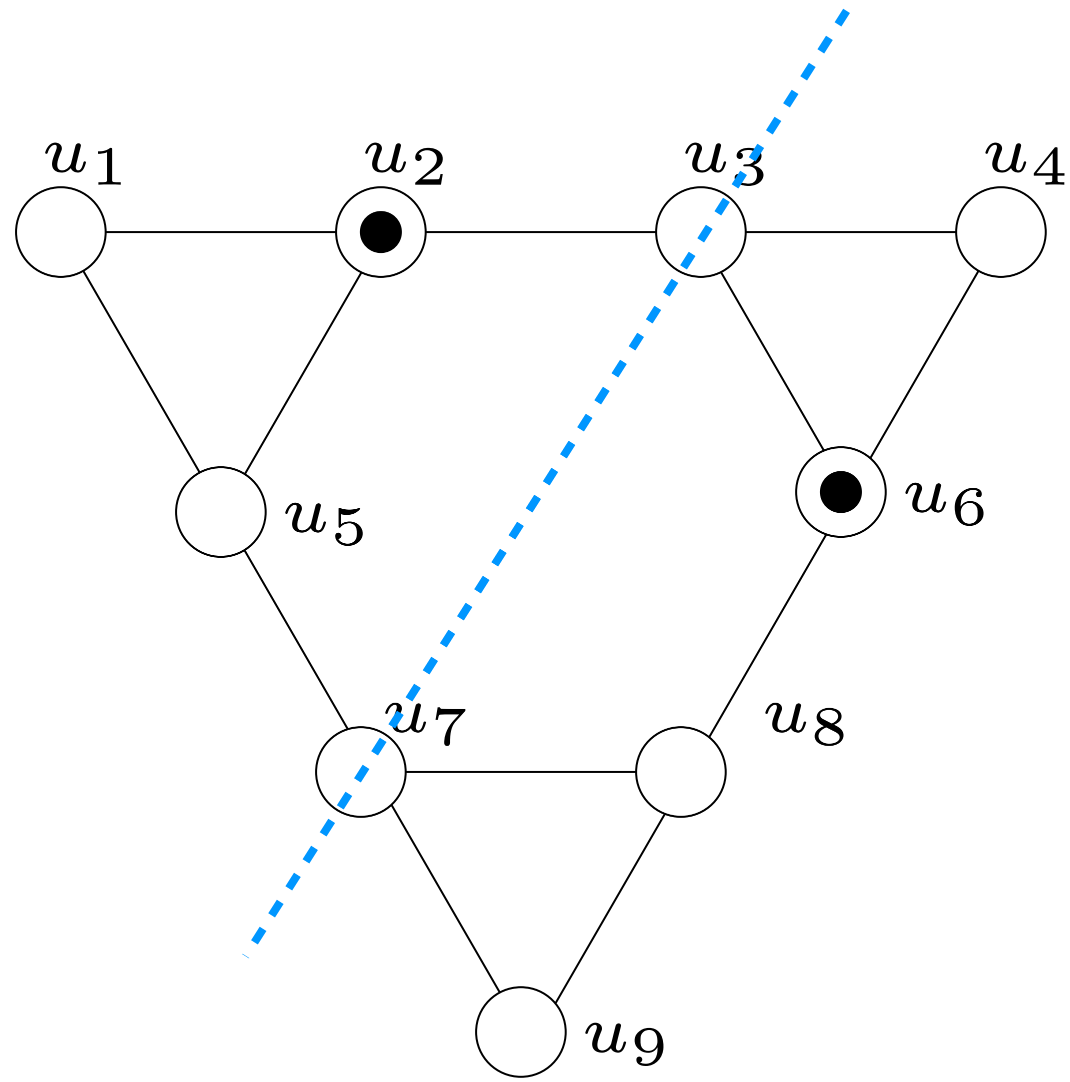
- Voronoi Games
- Matrix Scheduling Games
- Connection Games

Gadget

Lemma: There is no Nash equilibrium with 2 players.

Proof: By sym., the first player choose u_2.
Then the second player moves to u_6 and gains 5.
Now the first player can move to u_7 to increase his utility.



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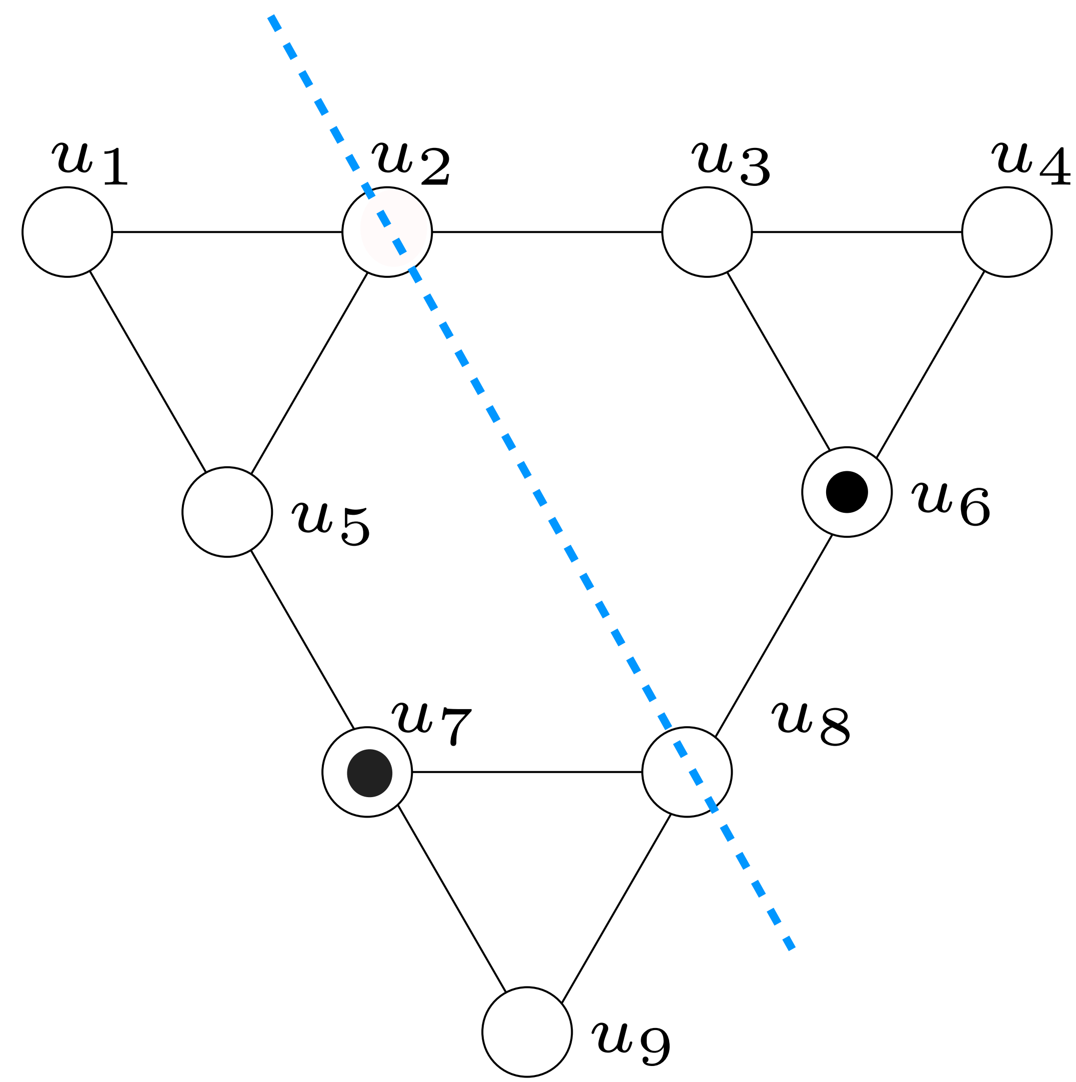

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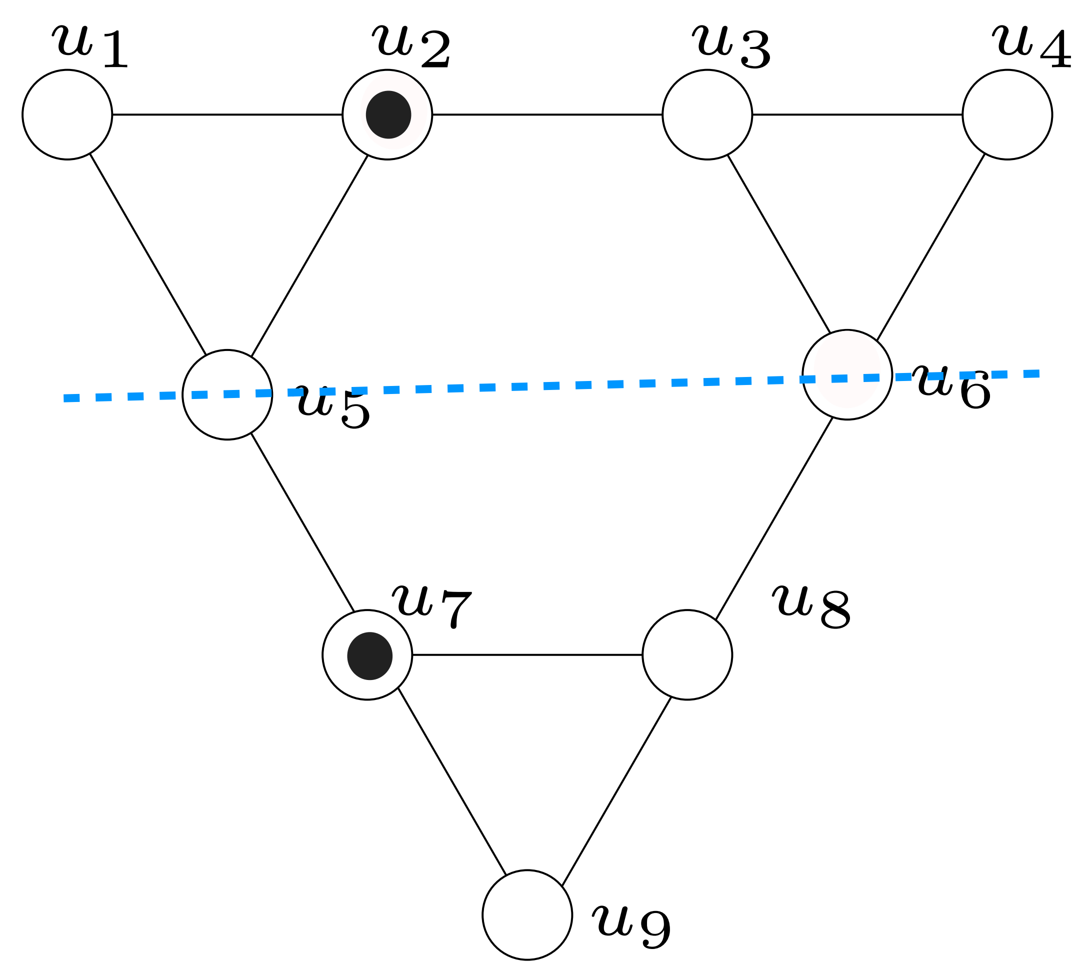

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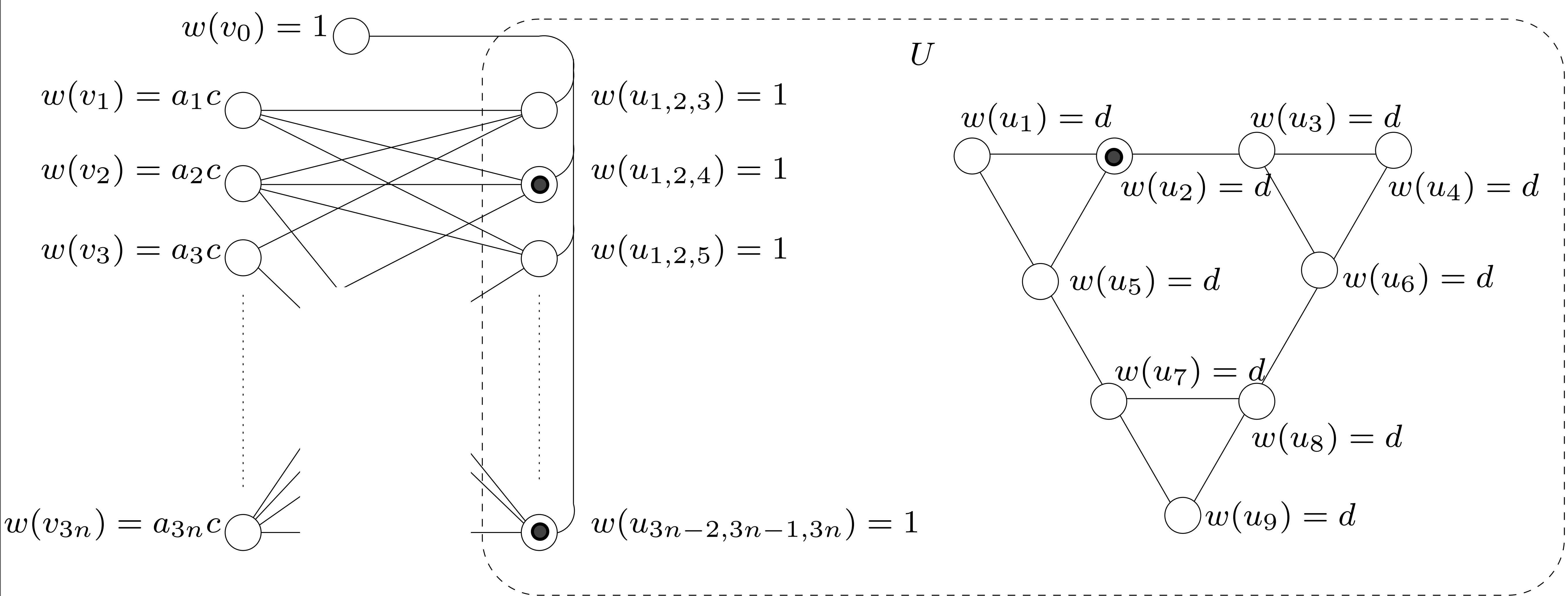
NP-hardness

Theorem: It is NP-hard to decide whether a Voronoi game admits an equilibrium.

Proof (high-level):



Inefficiency

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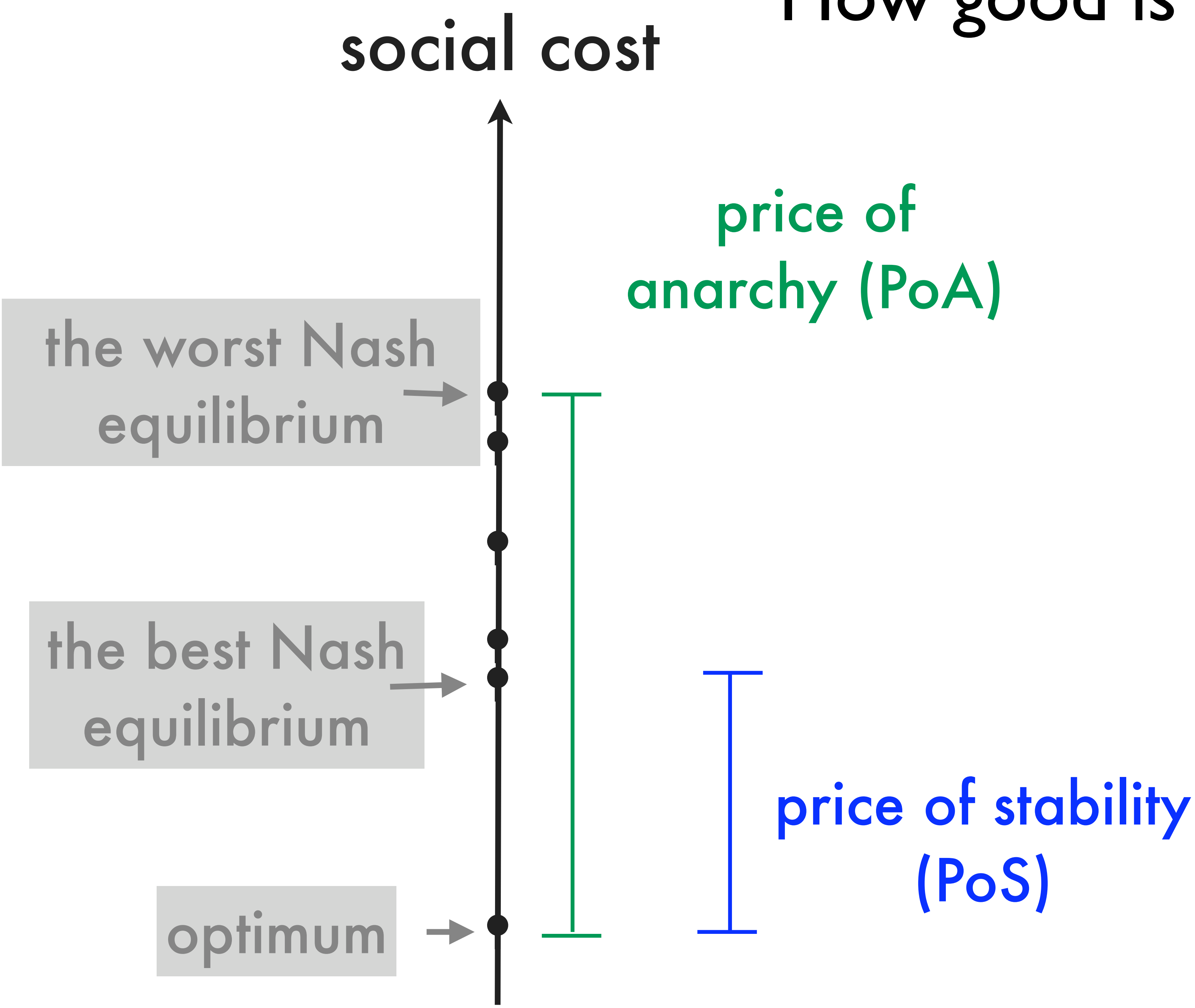

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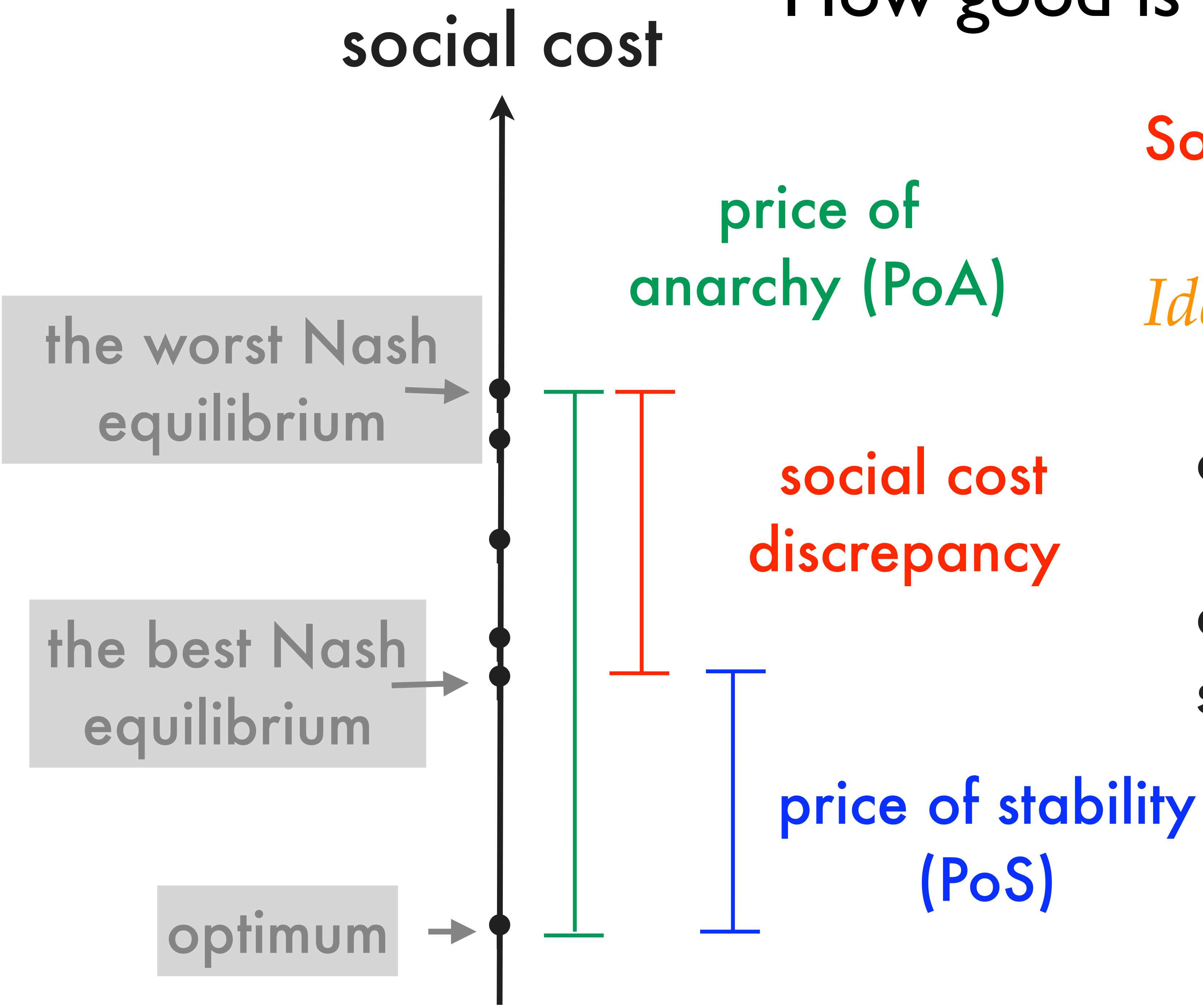

Social cost discrepancy:
worst Nash / best Nash

Ideas:

- Measure the degree of choice in a game.
- Unfair to compare the cost with OPT in selfish setting.

Delaunay triangulation

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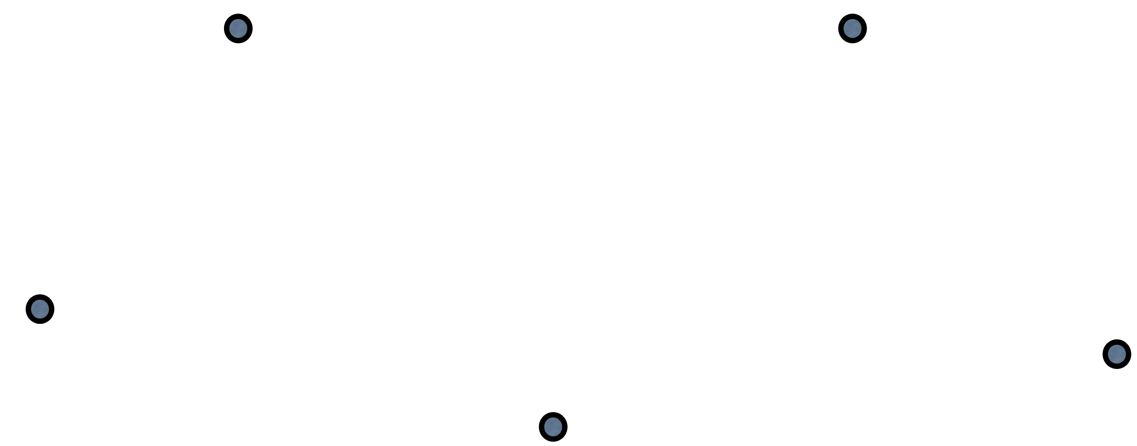
- Delaunay triangulation:



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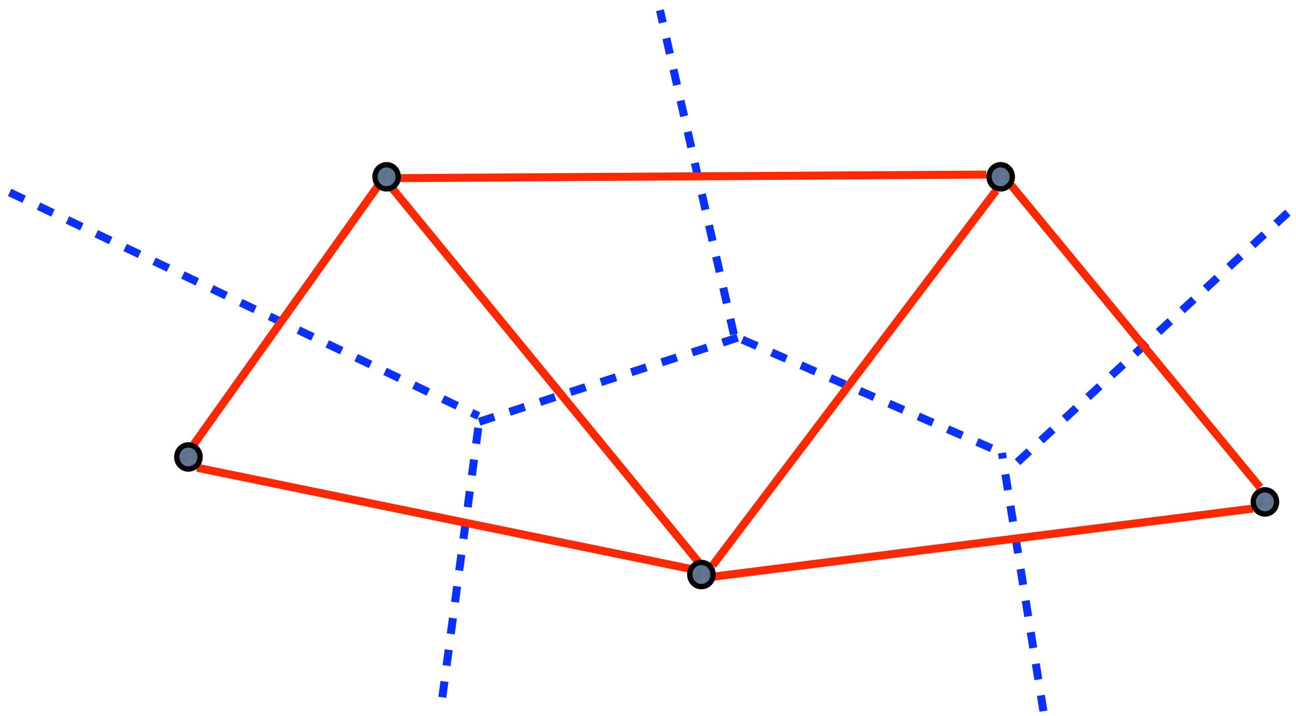

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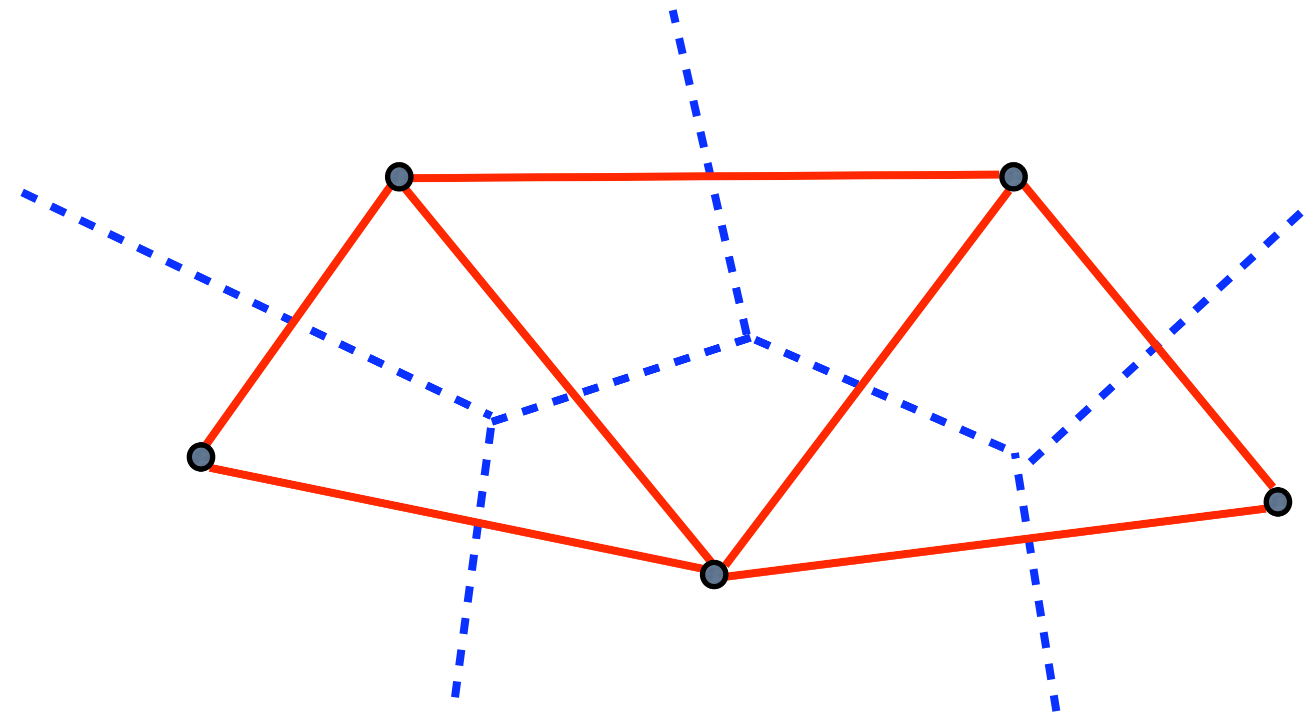
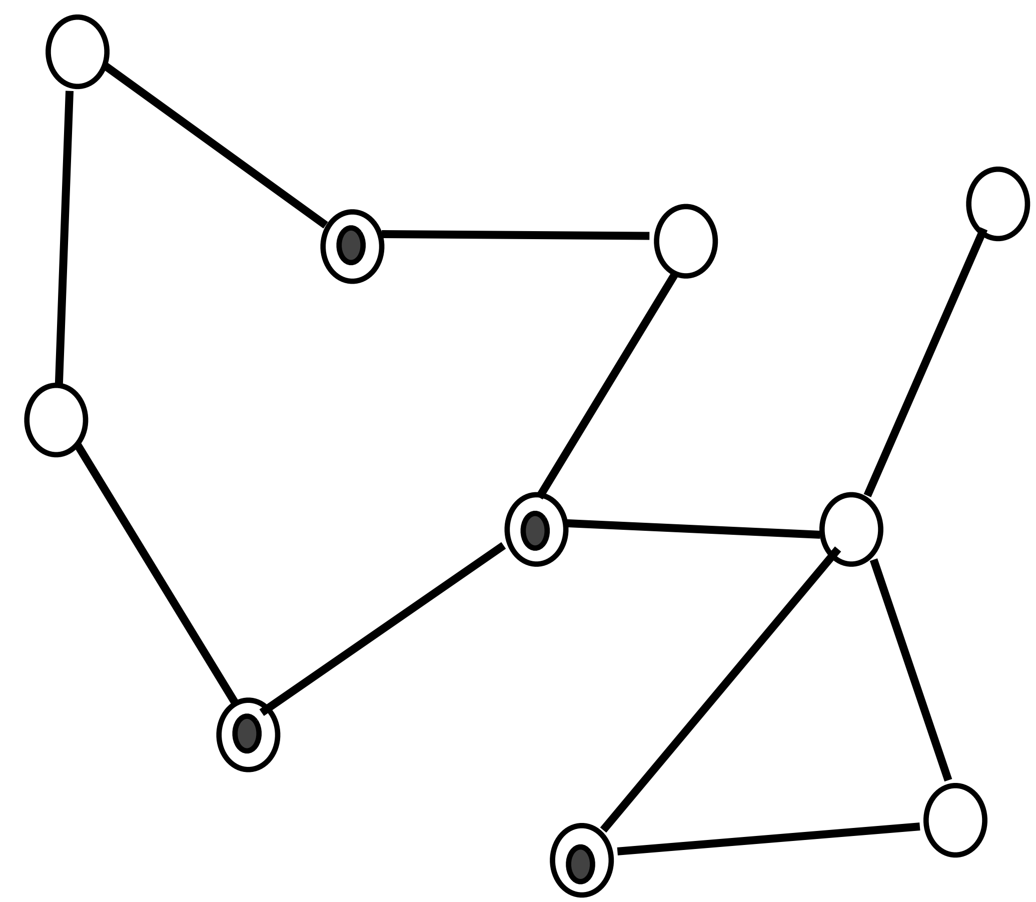


* Delaunay graph: a strategy profile f, there exists an edge (i, j) if i, j are neighbors.



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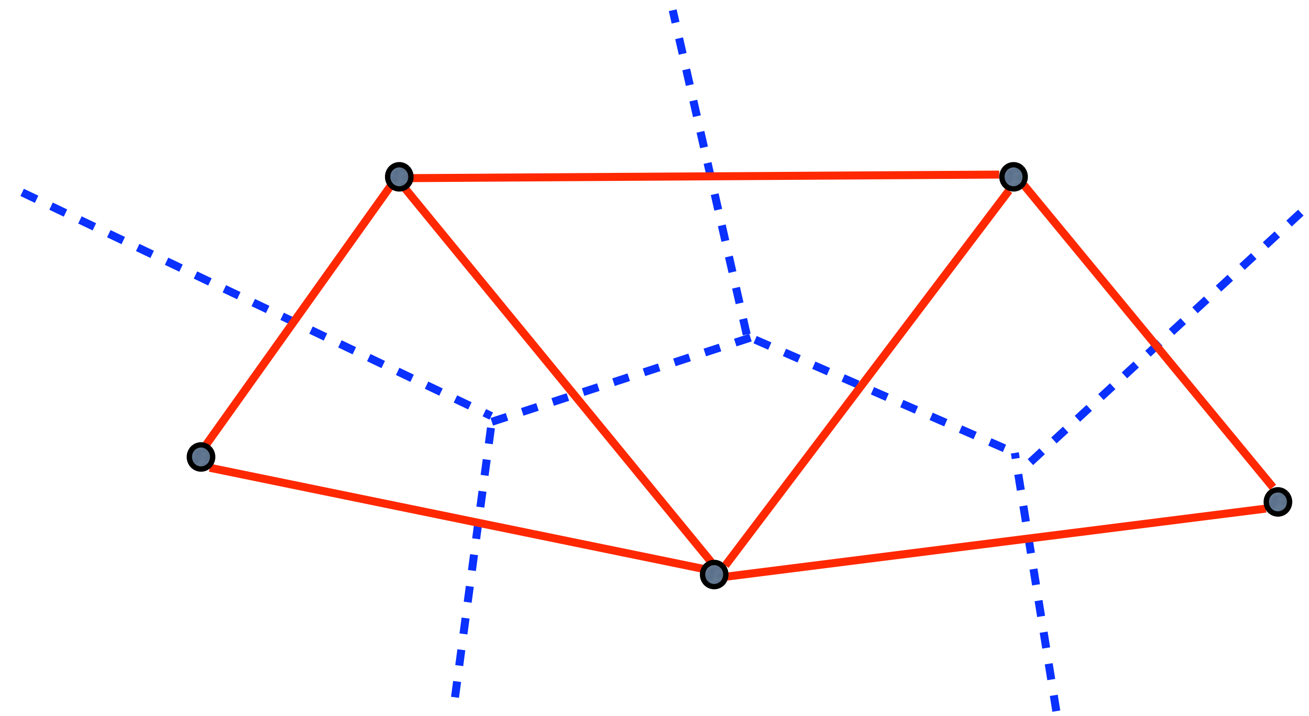

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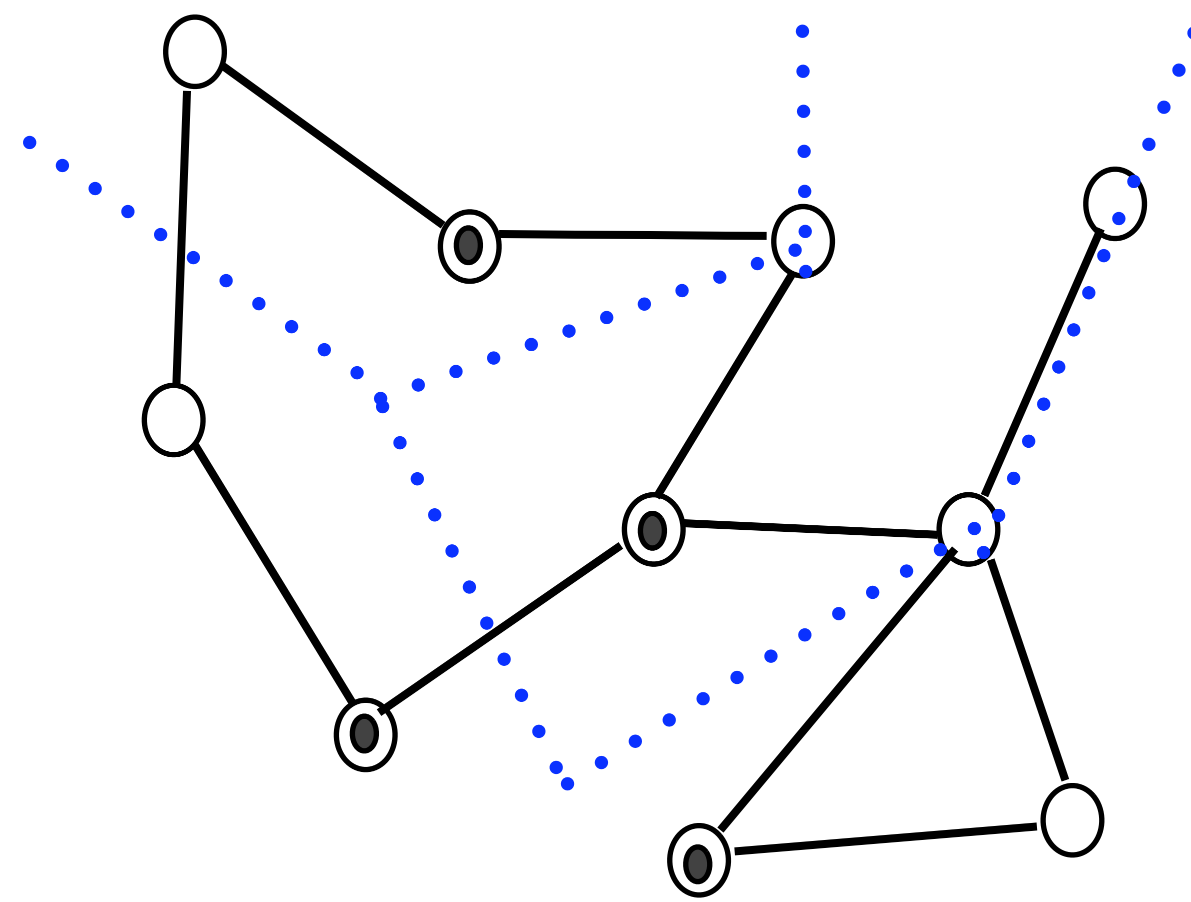

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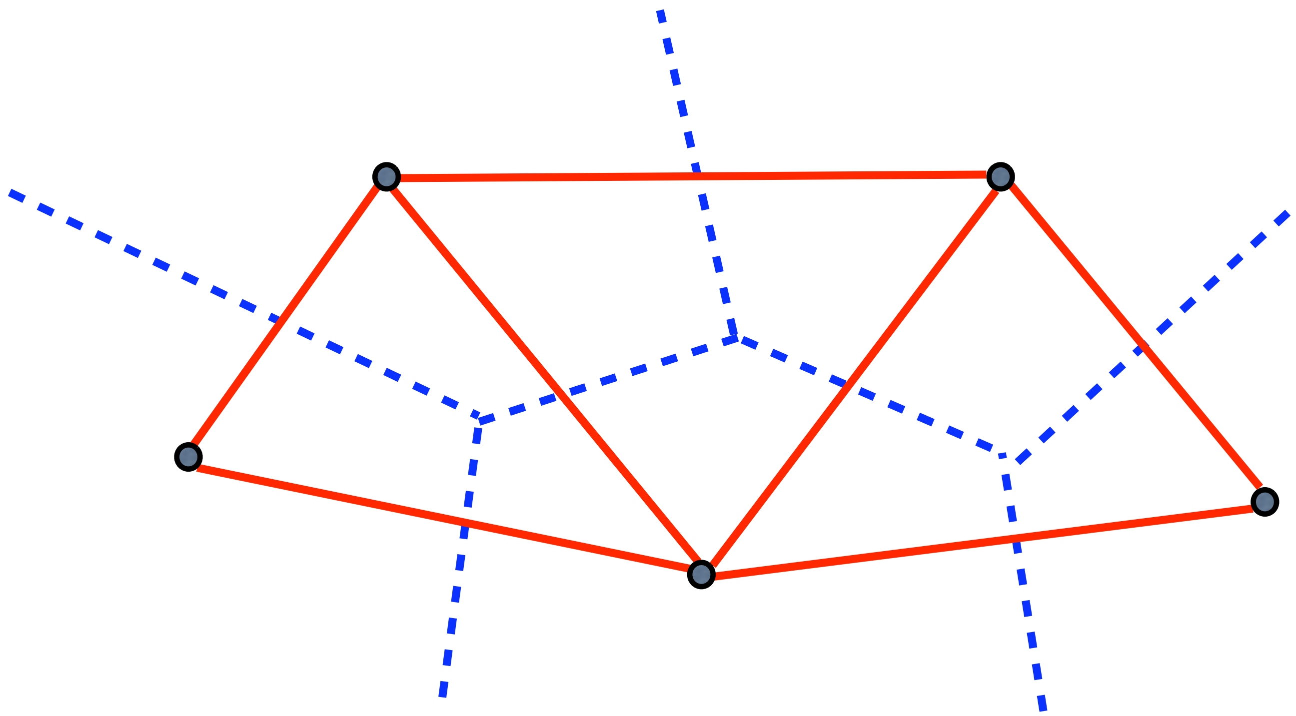

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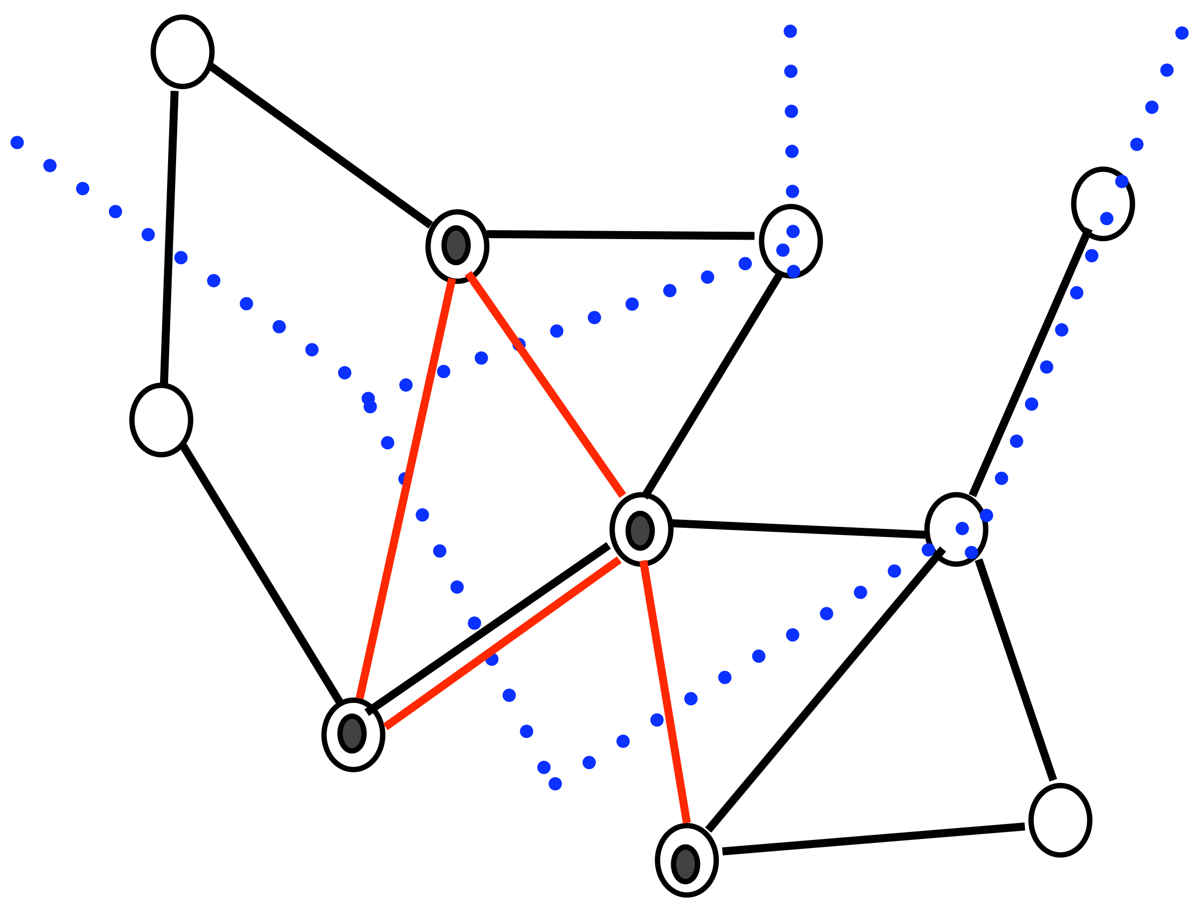
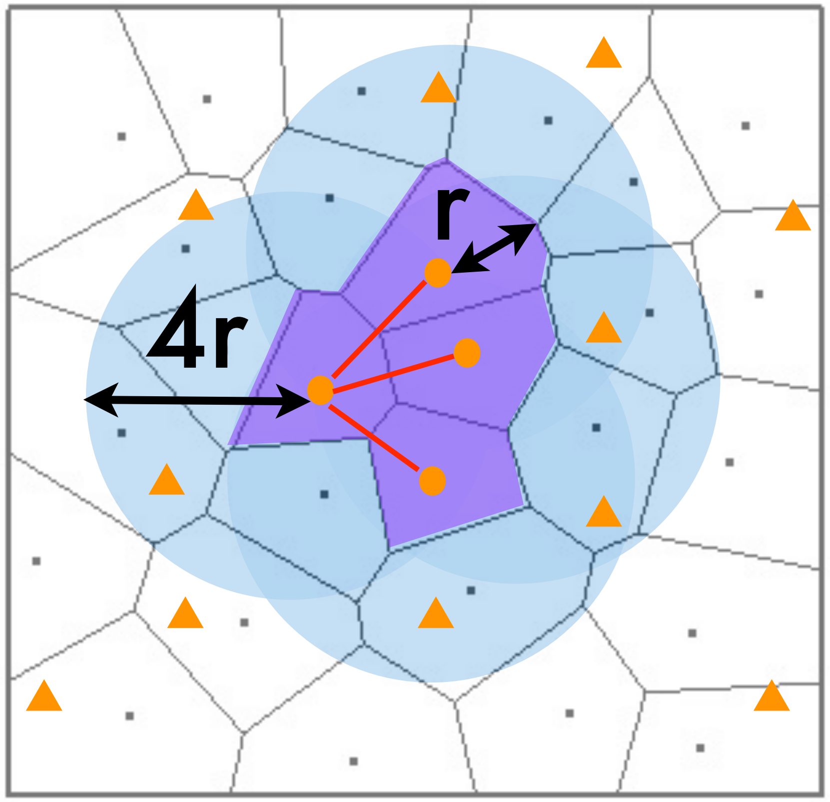


Social cost discrepancy

Theorem: The social cost discrepancy is $\Omega(\sqrt{n/k})$ and $O(\sqrt{kn})$

Proof:

- Consider two equilibria ● and ▲.
- Partition the Delaunay graph corresponding to ● into regions.
- Showing that each location of ▲ is not so far from a region above (compared to the diameter of the region).



Improvements

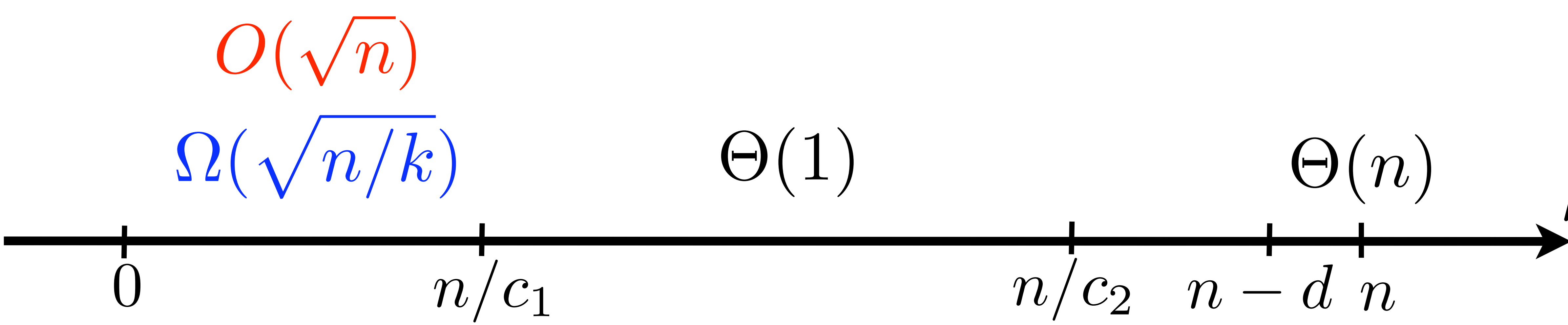
Theorem:

- If $k \leq n/4$ then the discrepancy is $O(\sqrt{n})$
- If there exist constants $c_1 \geq c_2, n_0$ such that:

  $\forall\ n \geq n_0 : n/c_1 \leq k \leq n/c_2$

  then the discrepancy is $\Theta(1)$
- If there exists constant d such that $k \geq n - d$

  then the discrepancy is $\Theta(n)$



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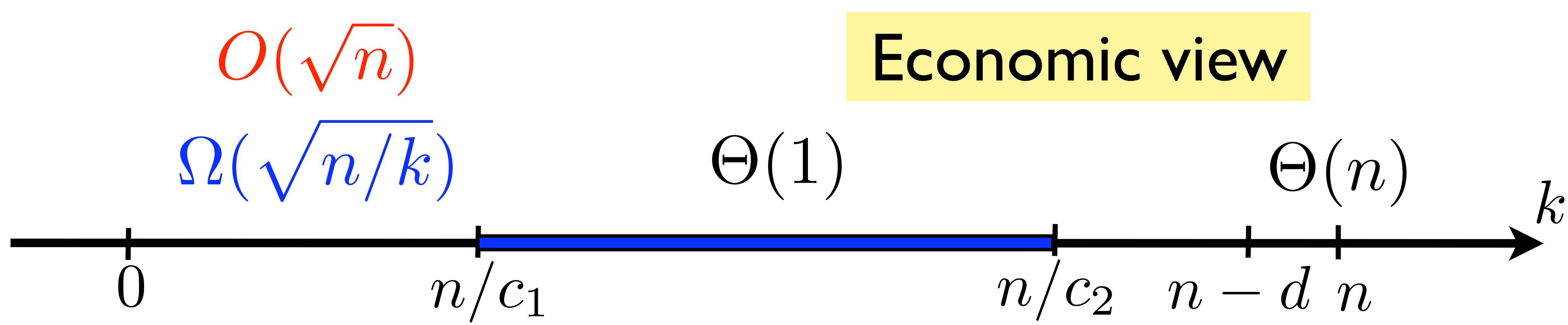

Scheduling Games in the Dark

Scheduling Games

- n jobs (players) and m machines: a job chooses a machine to execute. The processing time of job i on machine j is p_{ij}
- The cost c_i of a job i is its completion time.
- The social cost is the makespan, i.e. $\max_i c_i$
- Each machine specifies a policy how jobs assigned to the machine are to be scheduled.

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Non-clairvoyant policies

existence
Nash equilibrium

small PoA

Natural policies

* RANDOM: schedules jobs in a random order.

  In the strategy profile σ, i is assigned to j:

$$c_i = p_{ij} + \frac{1}{2} \sum_{i' : \sigma(i') = j, i' \neq i} p_{i'j}$$

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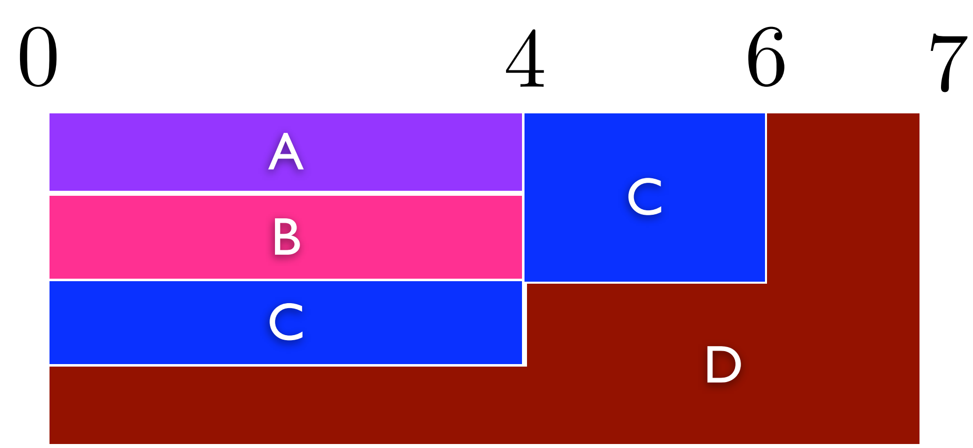
$$p_A = 1 \quad p_B = 1 \quad p_C = 2 \quad p_D = 3$$



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  If there are k jobs on machine j s.t: $p_{1j} \leq \ldots \leq p_{kj}$

$$c_i = p_{1j} + \ldots + p_{i-1,j} + (k - i + 1) p_{ij}$$

Definitions

* Def: A job i is balanced if $\max p_{ij} / \min p_{ij} \leq 2$

* Def of models:
  - Identical machines: $p_{ij} = p_i \; \forall j$ for some length p_i
  - Uniform machines: $p_{ij} = p_i / s_j$ for some speed s_j
  - Unrelated machines: p_{ij} arbitrary

Definitions

* Def: A job is unhappy if it can decrease its cost by changing the strategy (other players' strategies are fixed)

* Def: Best-response dynamic is a process that let an arbitrary unhappy player (job) make a best response -- a strategy that maximizes player's utility.

Our results

* Existence of equilibria: potential argument.

	identical	*uniform*	*unrelated*
RANDOM	*NE*		*non-convergence*
EQUI	*NE*	*NE*	*NE*

Idea:
- Best-response dynamic may cycle
- New dynamic to break the cycle.

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- Jobs have length $p_1 \leq p_2 \leq \ldots \leq p_n$
- Machines have speed $s_1 \geq s_2 \geq \ldots \geq s_m$

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* Dynamic: among all unhappy jobs, let the one with the greatest index make a best move.

Potential function

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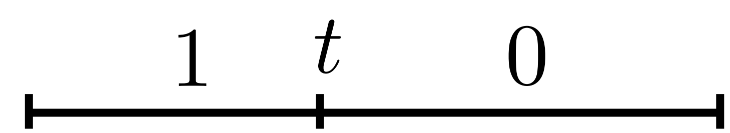
* For any strategy profile σ, let t be the unhappy job with greatest index.

$$f_\sigma(i) = \begin{cases} 1 & \text{if } 1 \leq i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$



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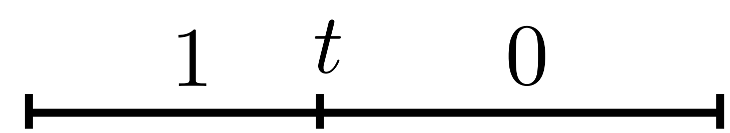

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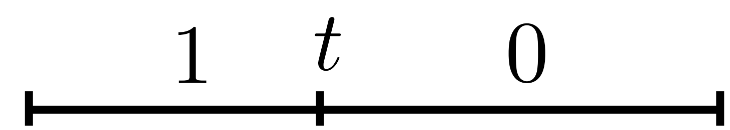

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* Dominance: either the number of unhappy players decreases or the lexicographical order of machines' speeds are decreased.

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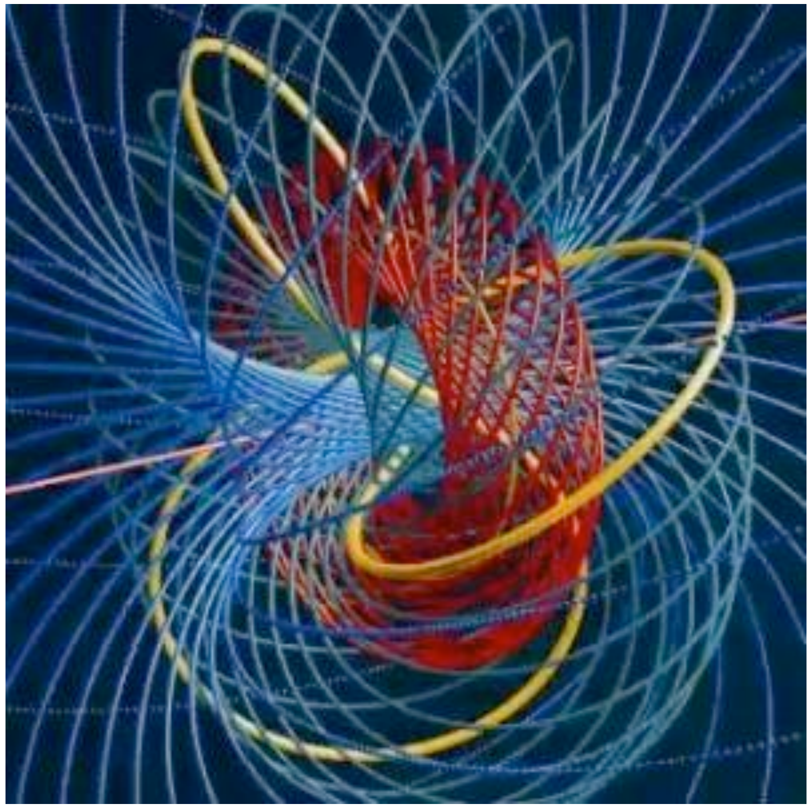
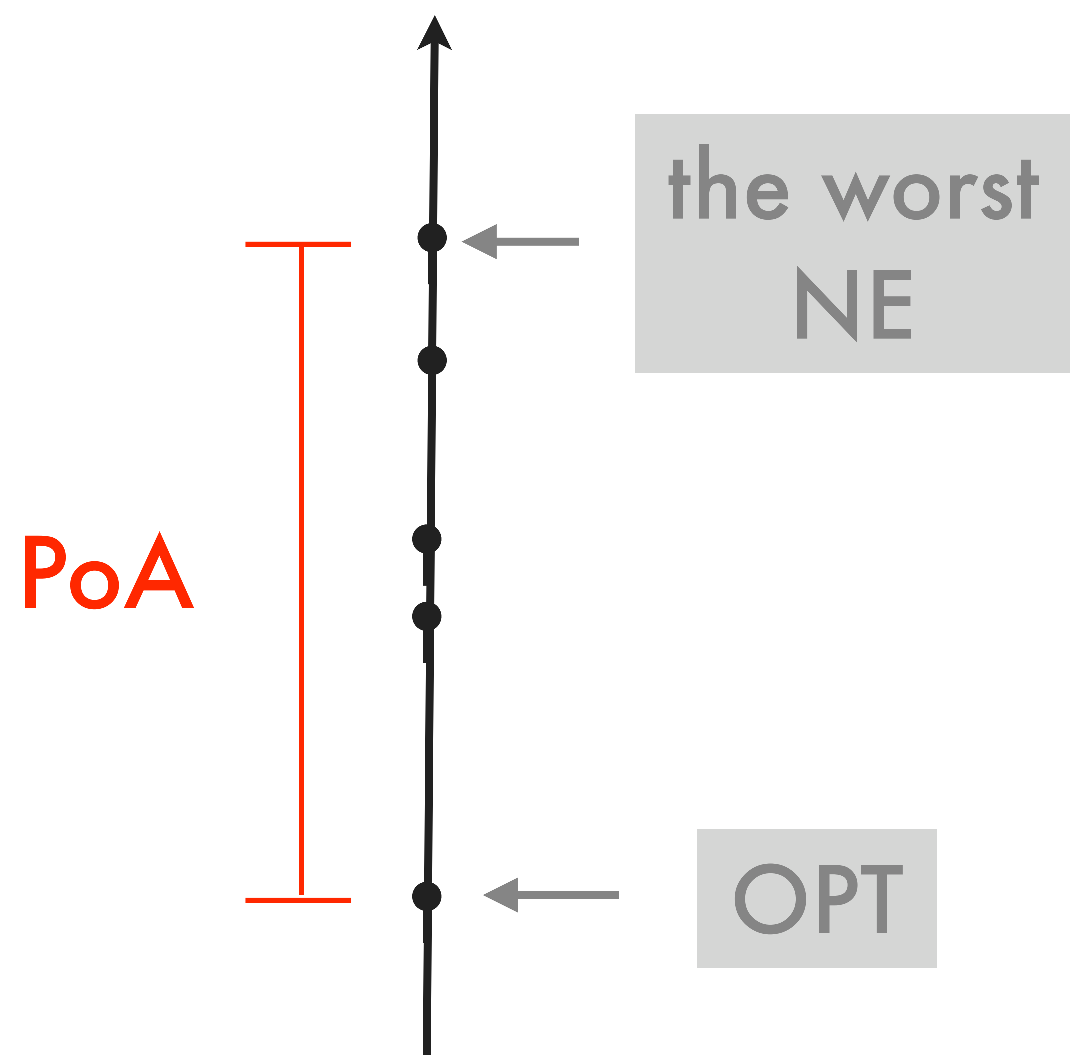


Our results

- *Theorem:* For unrelated machines, the PoA of policy EQUI is at most 2m – interestingly, that matches the best clairvoyant policy.

- PoA is not increased when processing times are unknown to the machines.

- The knowledge about jobs' characteristics is not necessarily needed.



Online Mechanism Design

Mechanism Design

Define the game

Goal: self-interested behavior yields desired outcomes.

Online Auction

* A company produces one perishable item per time unit (items have to be immediately delivered to bidders, e.g. electricity, ice-cream, ...)

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* Mechanism design:
  - w_i are private
  - Bidders may misreport their value. They bid b_i

Truthful Auction Design

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Auction:
receives all bids

allocation algorithm:
determine the set of satisfied bidders

payment algorithm:
determine how much a bidder has to pay

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$$u_i = \begin{cases} w_i - p_i & \text{if satisfied,} \\ 0 & \text{otherwise.} \end{cases}$$

Goal: self-interested behavior yields **truthfulness**, $b_i = w_i$

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payment algorithm:
determine how much a bidder has to pay

monotone:
a winner still wins if he raises his bid

critical bid:
the smallest bid that a winner needs to bid in order to win.

Monotone algorithm

- Our problem:
  - design a monotone allocation algorithm
  - verify whether the critical payment scheme can be computed efficiently.

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- Scheduling problem: $1|r_i - online,\ pmtn|\sum_i w_i$

  ... with monotone algorithm

Online Algorithm

* Def: an online algorithm ALG is c-competitive if for any instance I, the outcome $c \cdot ALG(I) \geq OPT(I)$

* Technique: charging scheme

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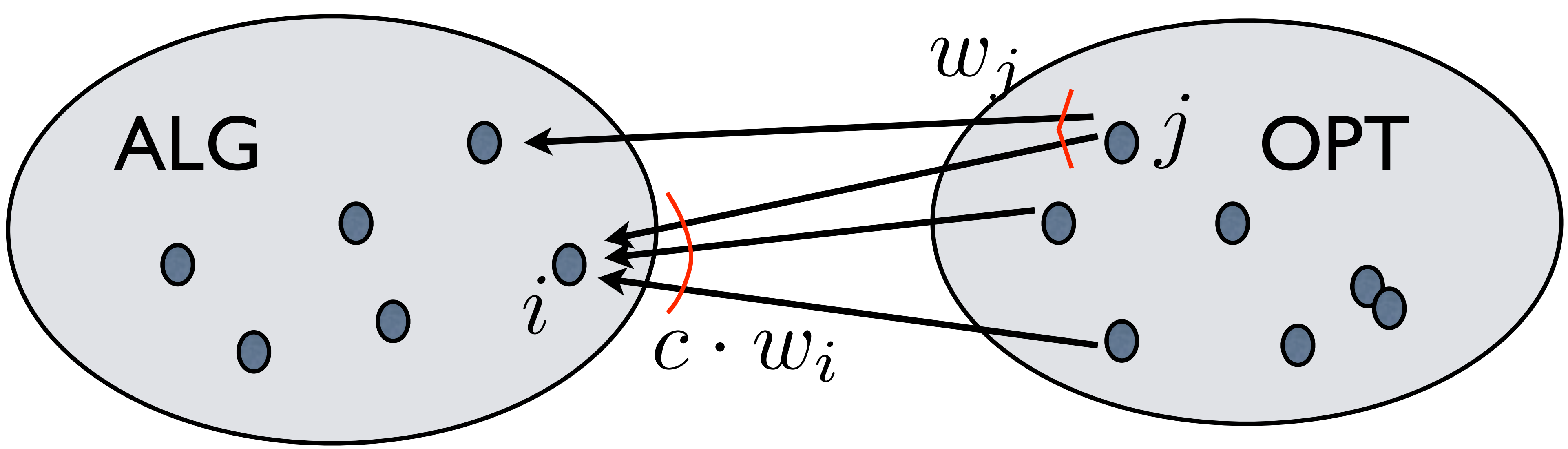

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  - The algo which serves the bidder that maximizes $b_i \cdot \alpha^{q_i - 1}$ is $\Theta(k/\log k)$-competitive where $\alpha = 1 - (1-\epsilon)^2 \cdot (\ln k)/k$.
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* *Proof:* Using general charging scheme.

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* *Corollary:* there exists truthful optimal mechanism with the same competitive ratio.

Summary

Given game

Players ⟷ Φ ⟷ Social objective

(maximize their utilities)

Summary

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Players
(maximize their utilities) $\longleftrightarrow$ Φ $\longleftrightarrow$ Social objective

X ?

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Design games (algorithms)
(a little more control,
much better quality of outcomes)

Thank you!