# NP-hardness of pure Nash equilibrium using negated gadgets

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# Equilibrium and Complexity

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Mixed equilibrium

Pure equilibrium

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choose a distribution over strategies

Pure equilibrium

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deterministically choose a strategy

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always exists (by Nash)

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Finding: PPAD-complete

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Finding: PLS-complete

Existence: NP-hardness

- Matrix Scheduling Games
- Connection Games

# Framework in proving NP-hardness

Negated gadget for property *P* of a game

+

A larger game which encodes a NP-hard problem

NP-hardness in deciding whether a game possesses property *P*

# Framework in proving NP-hardness

"counter example"

Negated gadget for property *P* of a game

+

A larger game which encodes a NP-hard problem

⟹ NP-hardness in deciding whether a game possesses property *P*

# Outline

- Connection Games / Weighted Connection Games
- NP-hardness:
  - equilibrium in Weighted Connection Games.
  - "good" cost-sharing protocol.
- Conclusion and open questions

# Connection Games

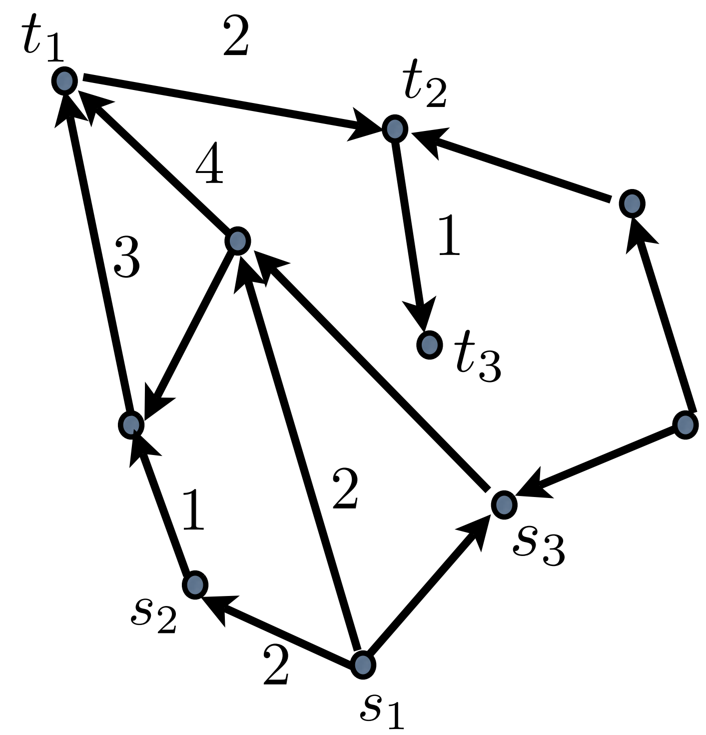
* $G(V, E)$ directed graph with cost $c : E \to \mathbf{Q}$

* $n$ players, each chooses (deterministically) a path $P_i$ to connect her source $s_i$ and sink $t_i$

* social cost: $\sum_{e \in \cup_i P_i} c_e$

* Shapley cost-sharing: cost of a player

$$\sum_{e \in P_i} c_e / n_e$$



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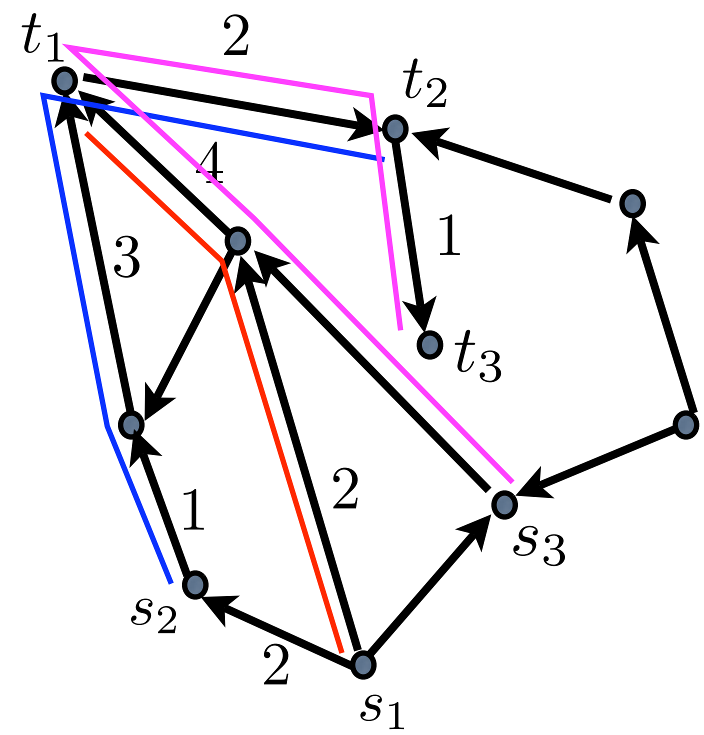

# Weighted Connection Games

- Similar to Connection Games but now each player has weight $w_i$

- weighted Shapley cost-sharing: cost of a player

$$\sum_{e \in P_i} c_e \cdot w_i / W_e \quad \text{where} \quad W_e = \sum_{j : e \in P_j} w_j$$

# Existence of equilibrium

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* The Connection Games always possesses a Nash equilibrium

$$\Phi(S) = \sum_{e \in S} \sum_{k=1}^{n_e} c_e/k, \quad \text{where} \quad S = \cup_i P_i$$

if a player *i* changes path $P_i$ by path $P_i'$

$$0 < \sum_{e \in P_i} c_e/n_e - \sum_{e \in P_i'} c_e/(n_e+1)$$

$$= \Phi(S) - \Phi(S \setminus P_i \cup P_i')$$

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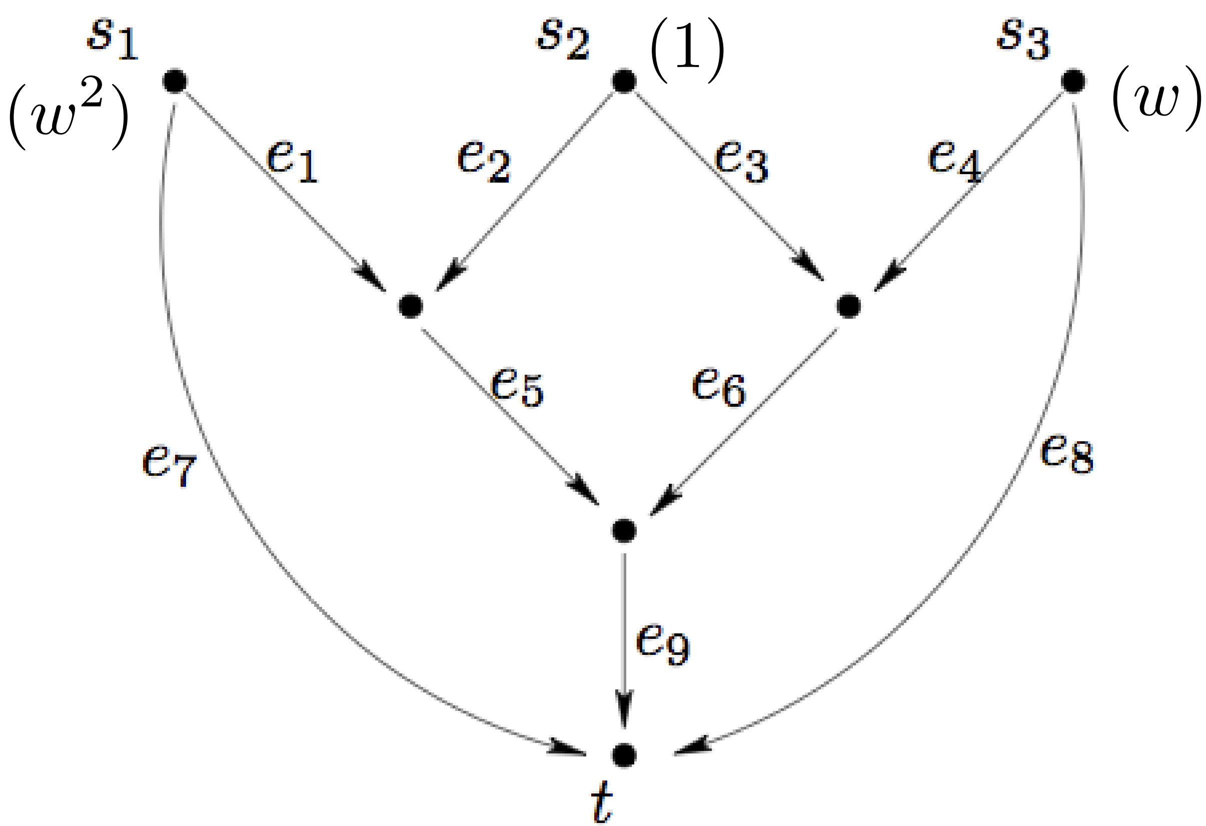
* *Lemma [Chen et al]*: There does not always exist Nash equilibrium in Weighted Connection Games.

# A counterexample

In the network, three players have weight $w^2, 1, w$

$$c_5 \cdot \frac{w^2}{w^2+1} + c_9 \cdot \frac{w^2}{w^2+1} > c_7 > c_5 + c_9 \cdot \frac{w^2}{w^2+w+1}$$

$$c_6 + c_9 \cdot \frac{w}{w^2+w+1} > c_8 > c_6 \cdot \frac{w}{w+1} + c_9 \cdot \frac{w}{w+1}$$



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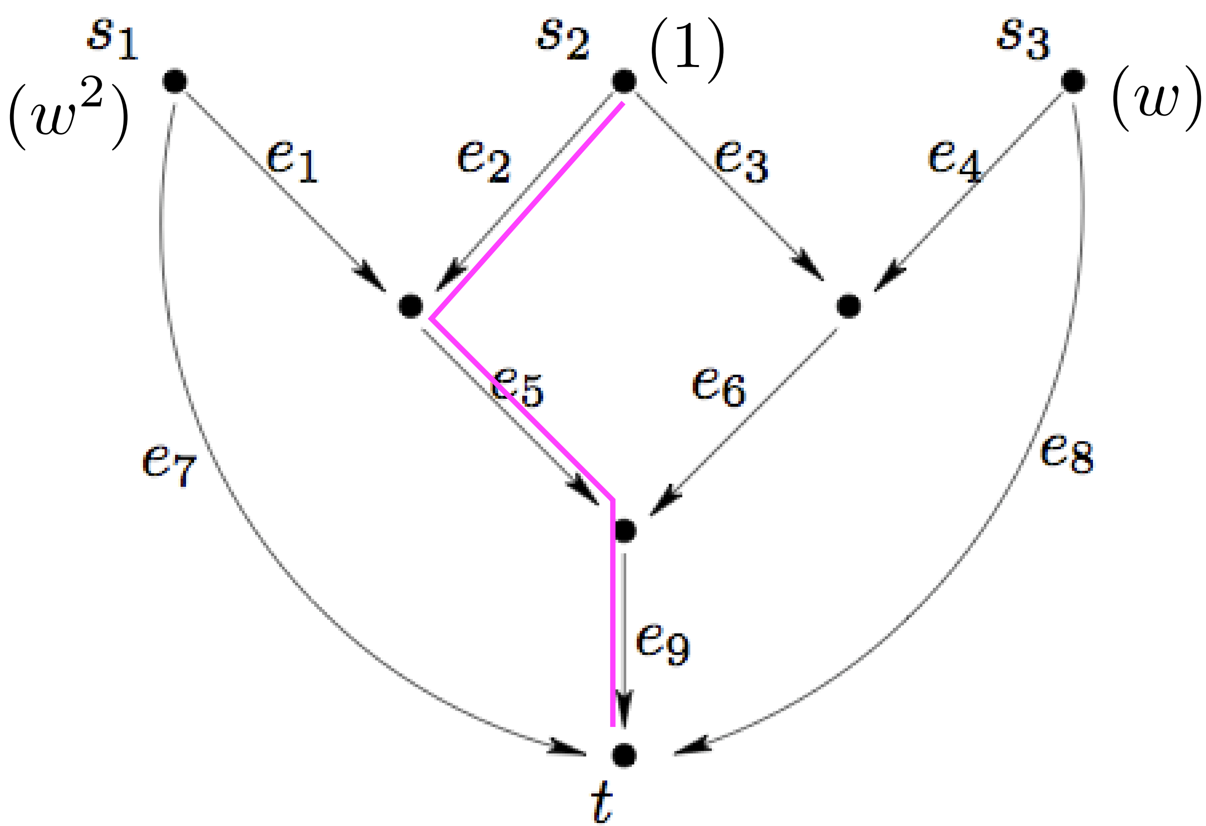

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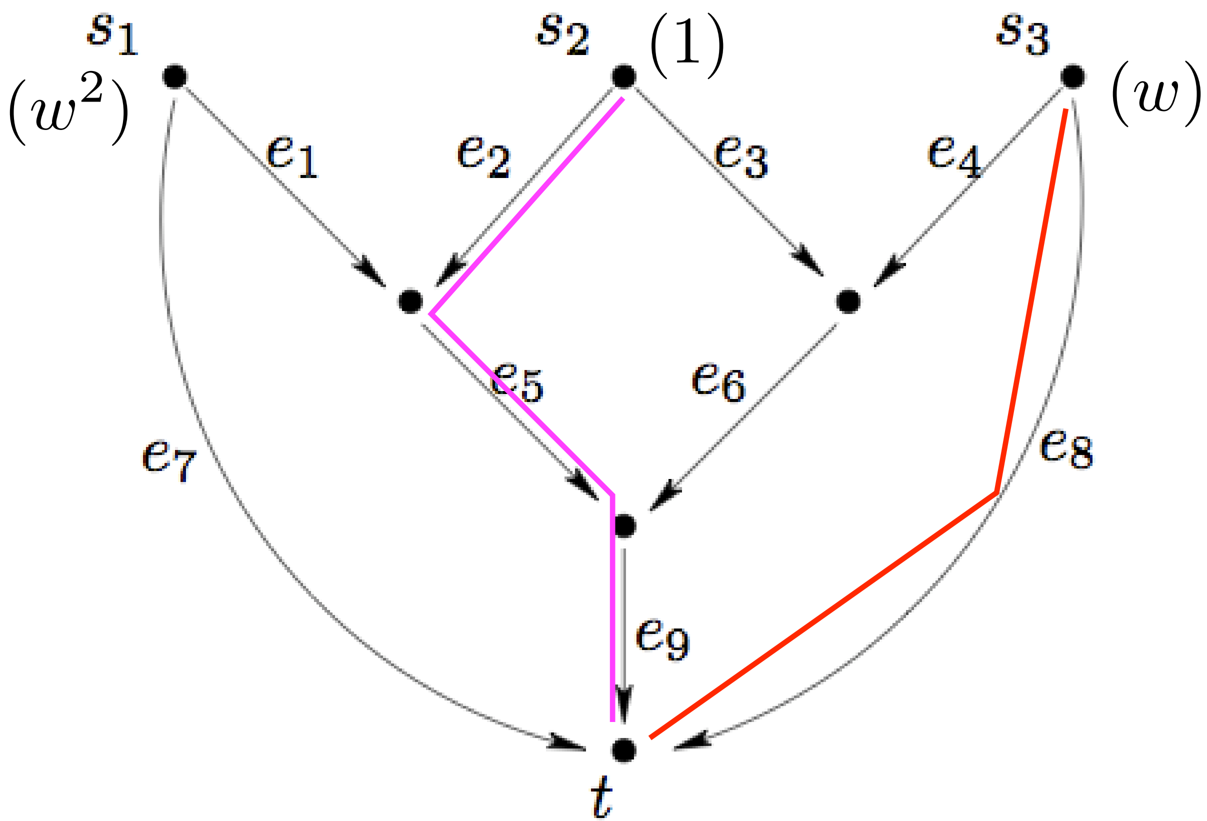

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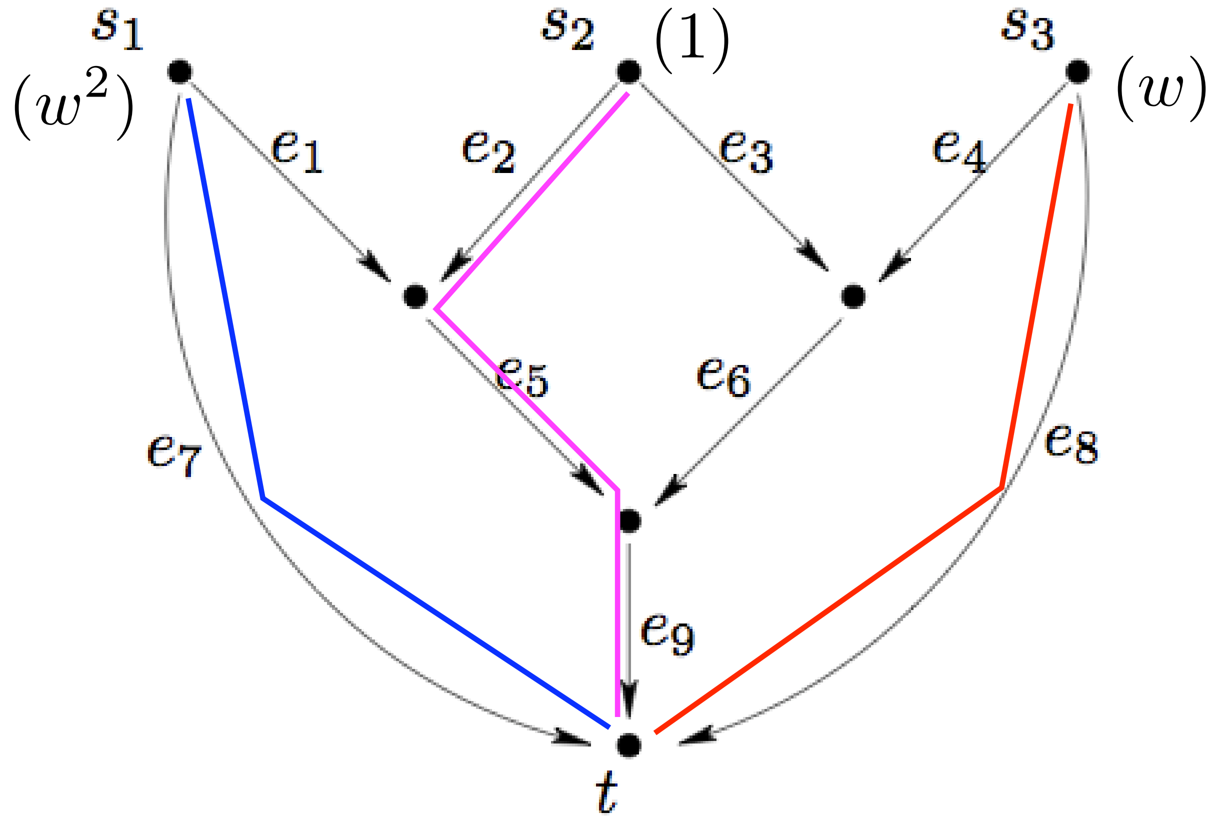

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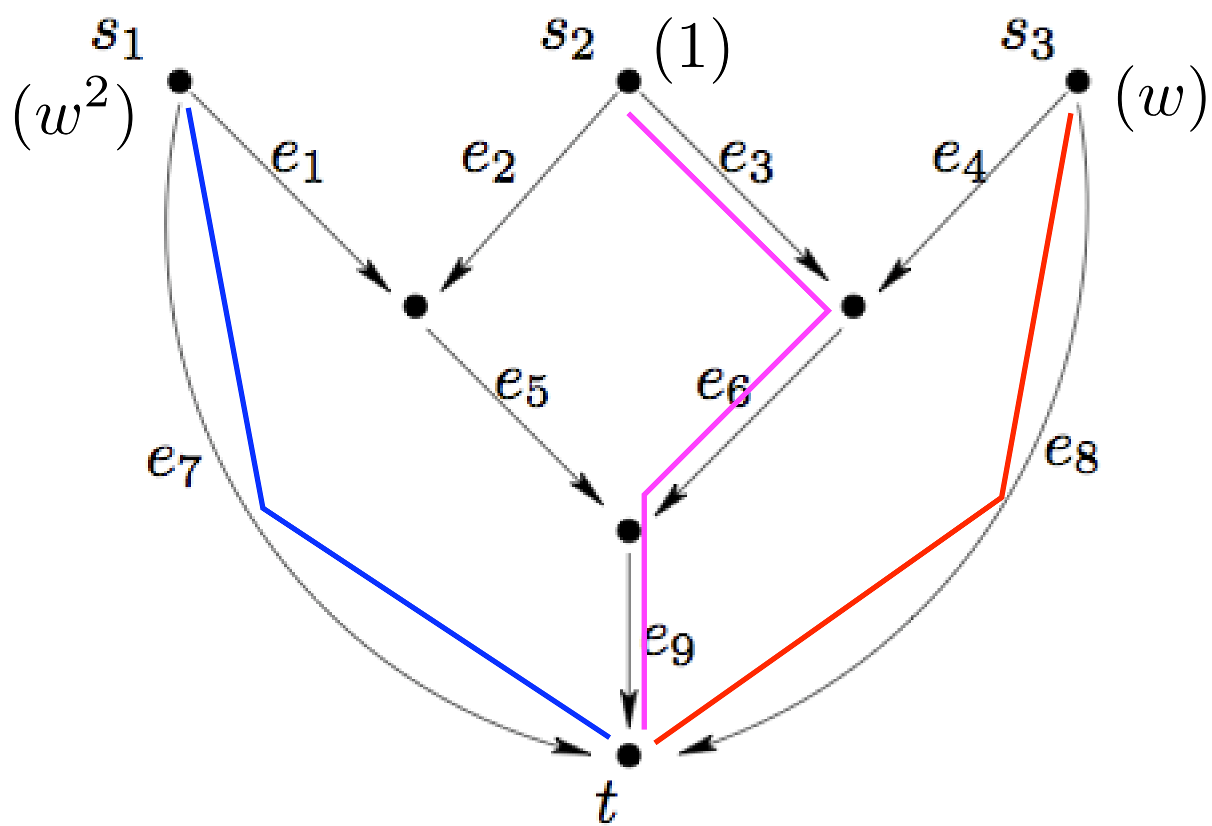

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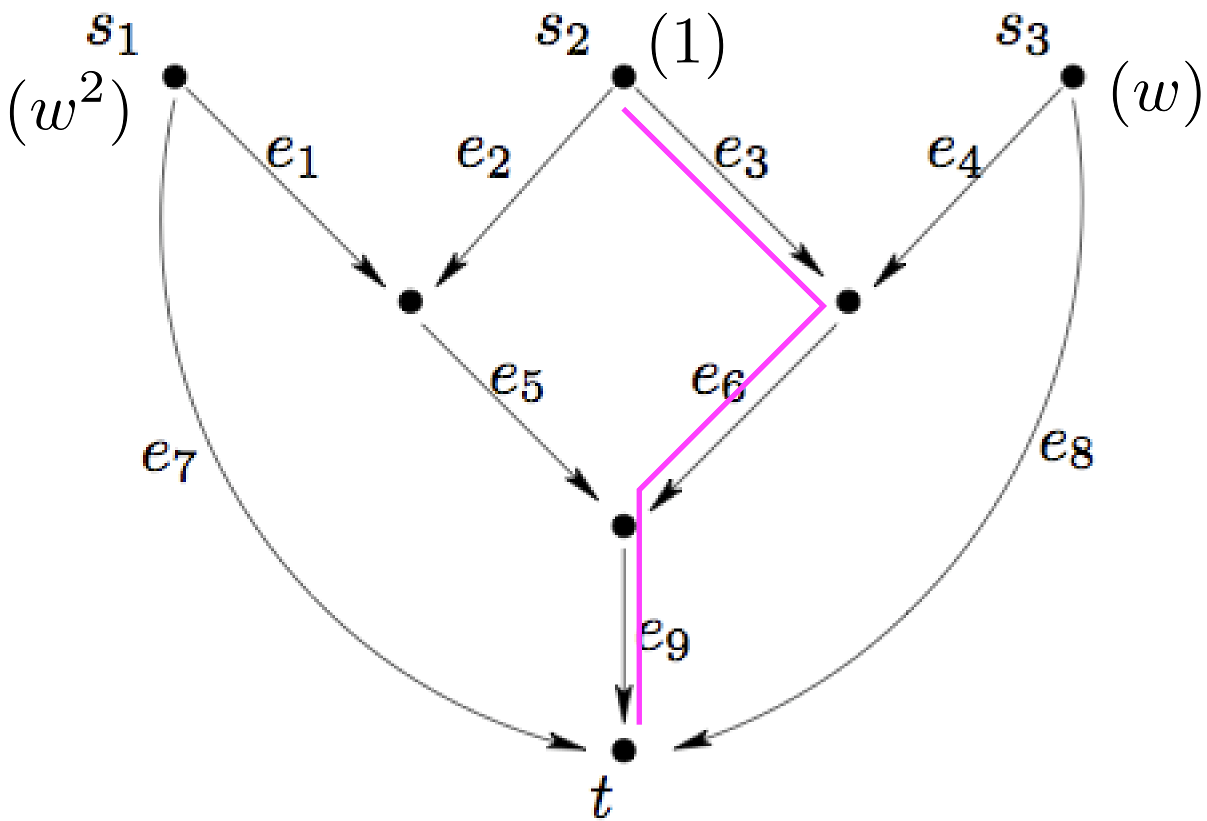

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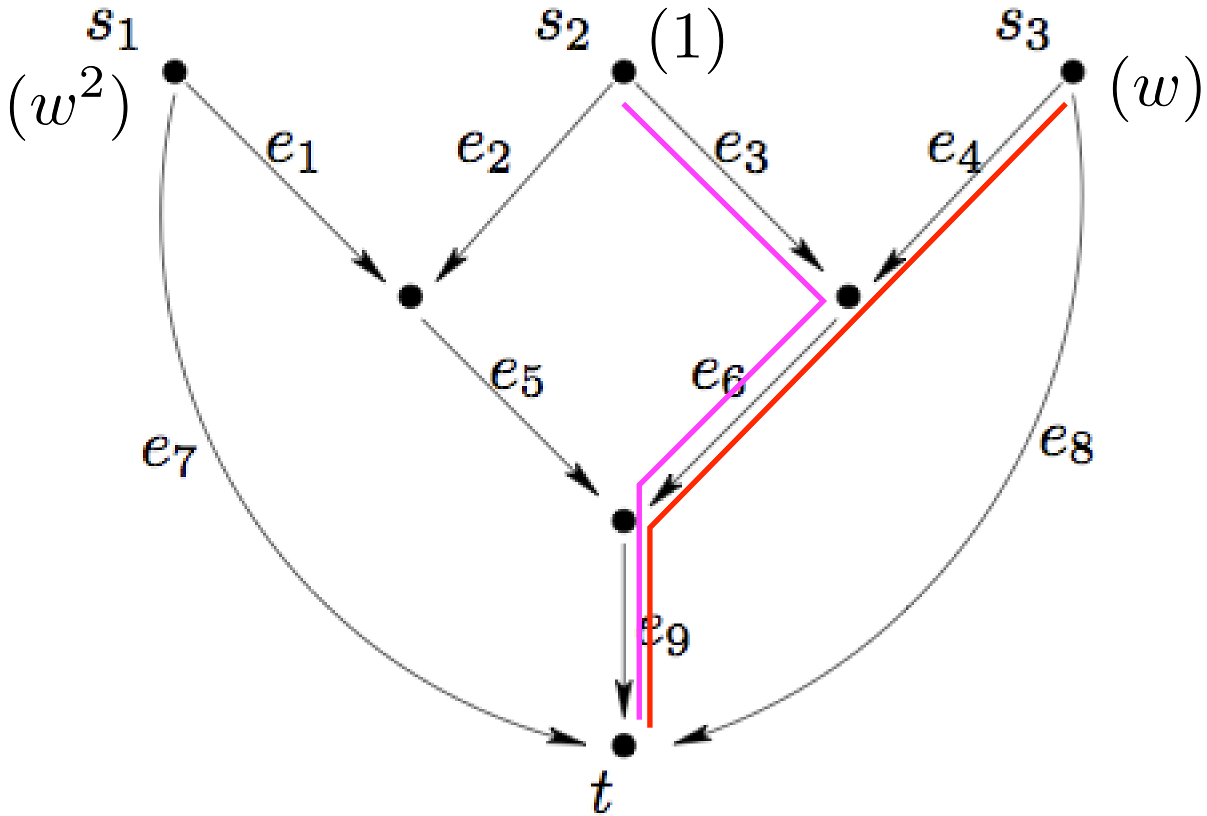

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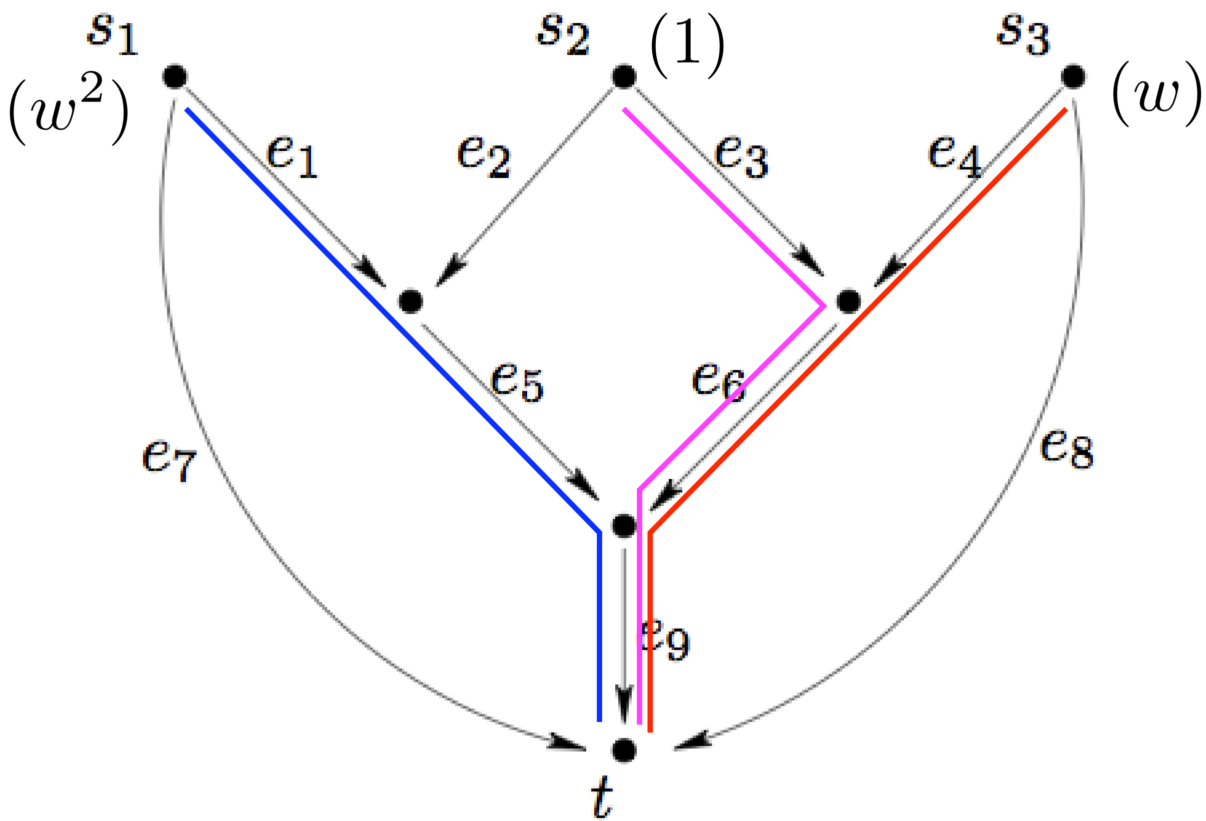

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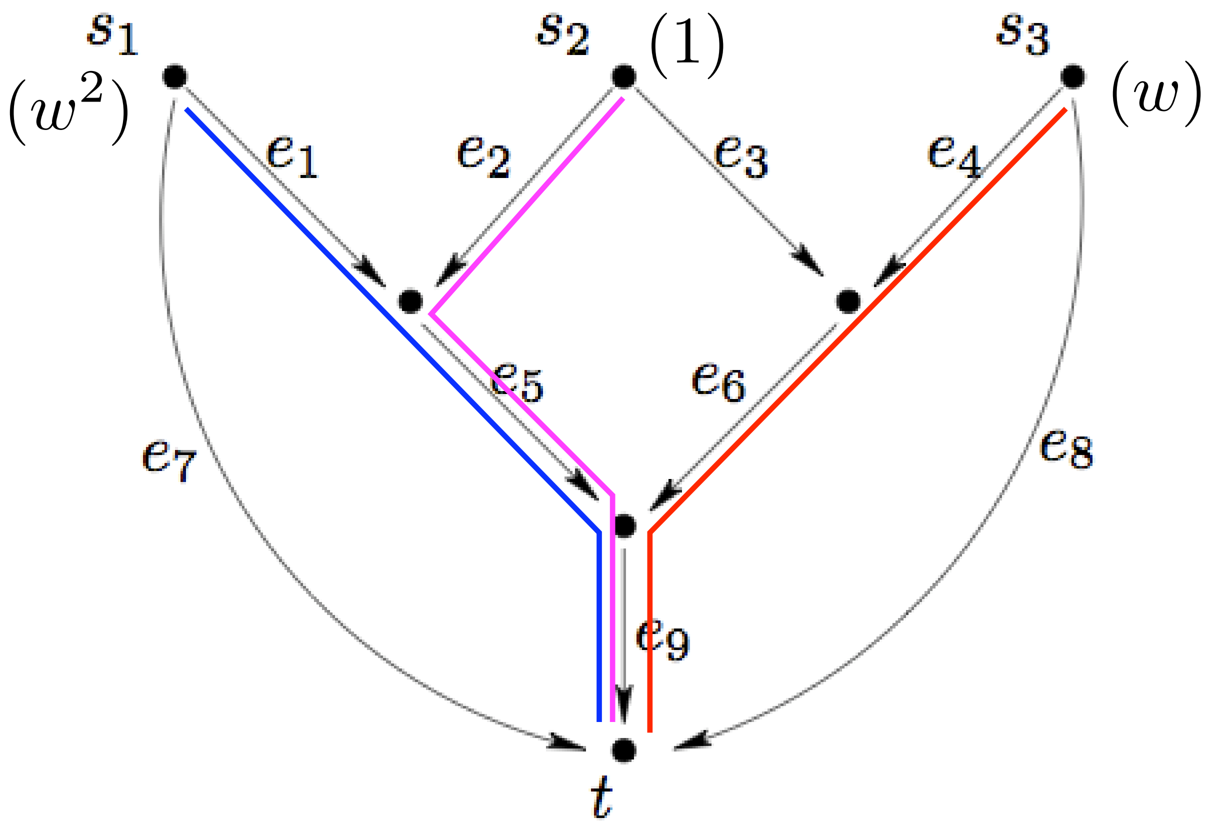


# NP-hardness

- *Theorem*: It is NP-hard to decide whether a given weighted connection game with Shapley cost-sharing admits an equilibrium.

- *Proof*: Reduction from MONOTONE3SAT.

MONOTONE3SAT: $X = \{x_1, x_2, \ldots, x_n\}$ $C = \{c_1, c_2, \ldots, c_m\}$

either $c = (x_1 \vee x_2 \vee x_3)$ or $c = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

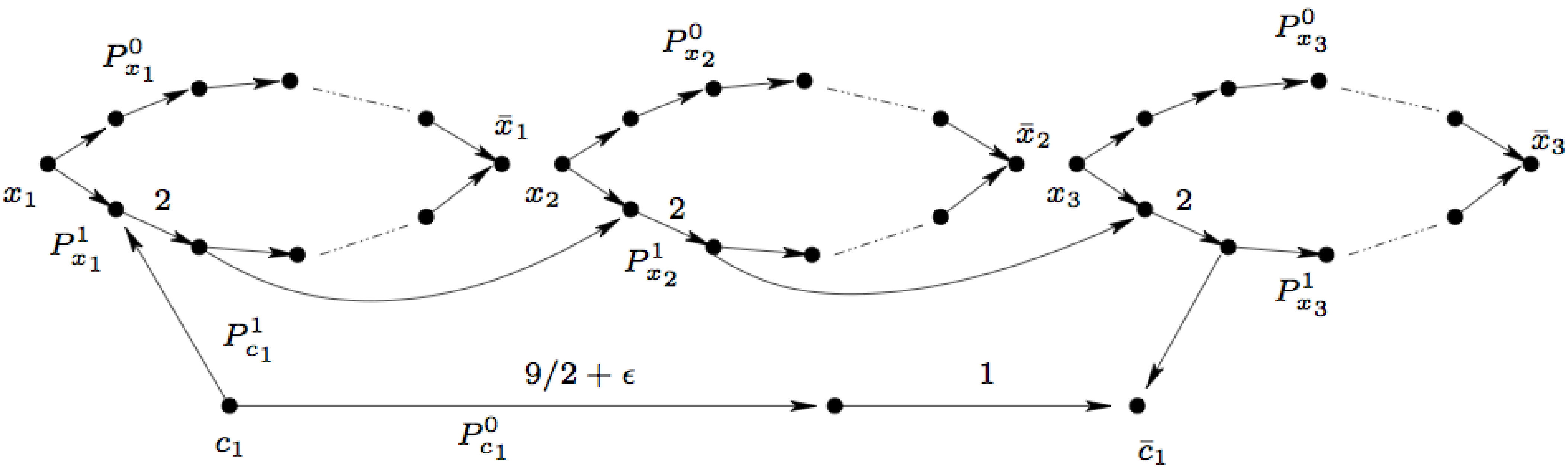
Network:

* Each player $p_x$ for a literal $x$ and player $p_c$ for a clause $c$
* Plug the gadget as a subnetwork
* Additional players

# Reduction Network

# Reduction Network

$$c_1 = x_1 \vee x_2 \vee x_3$$



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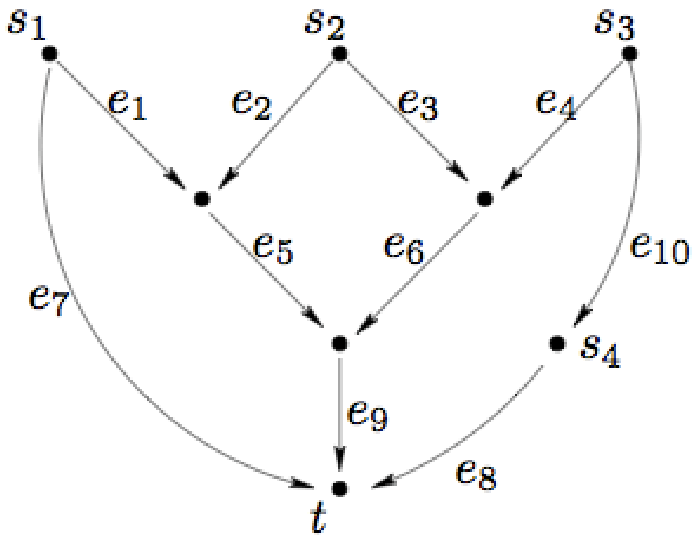


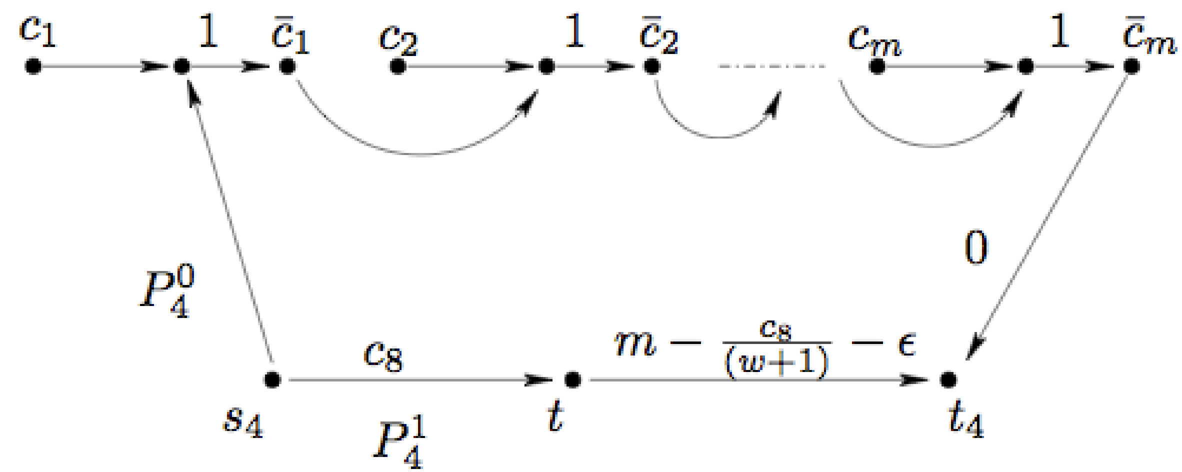


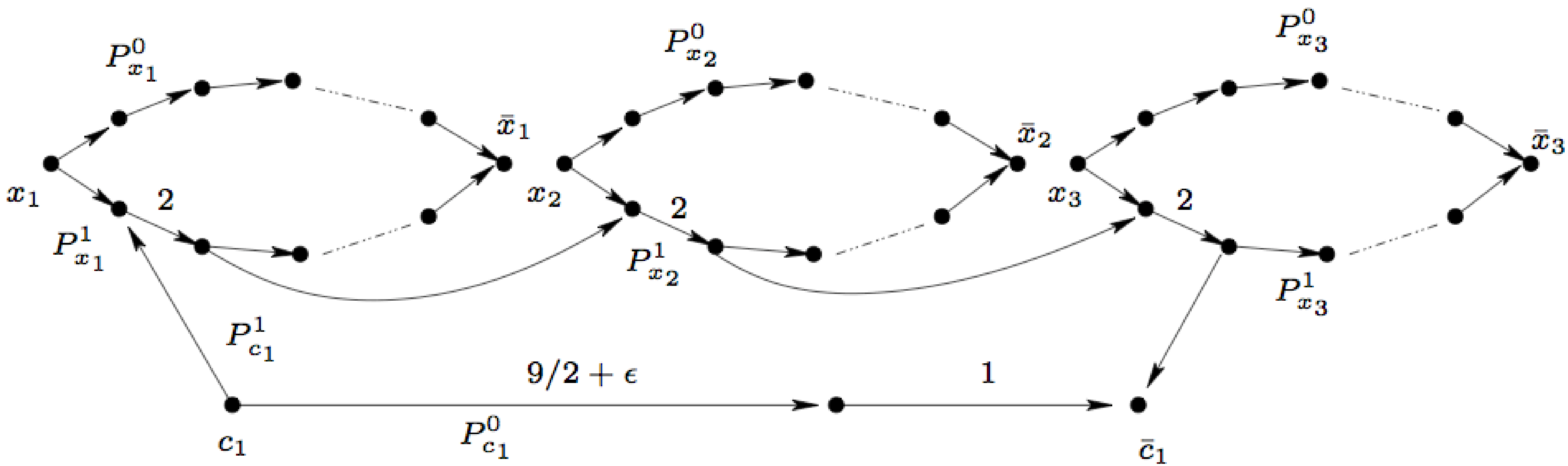

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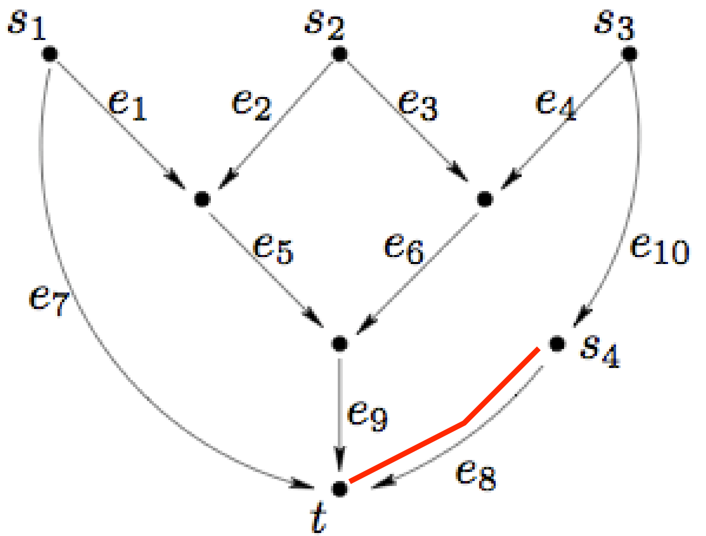


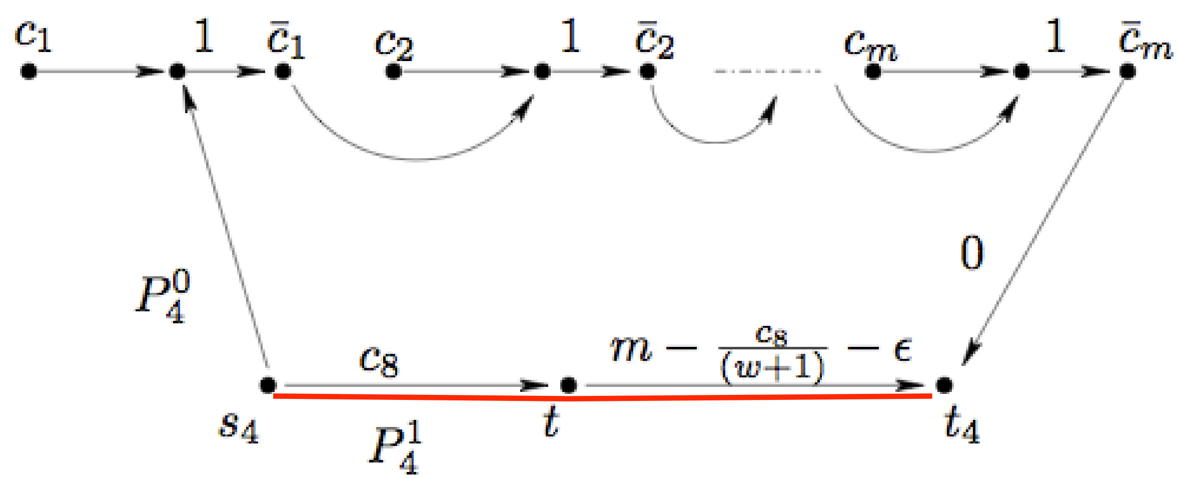


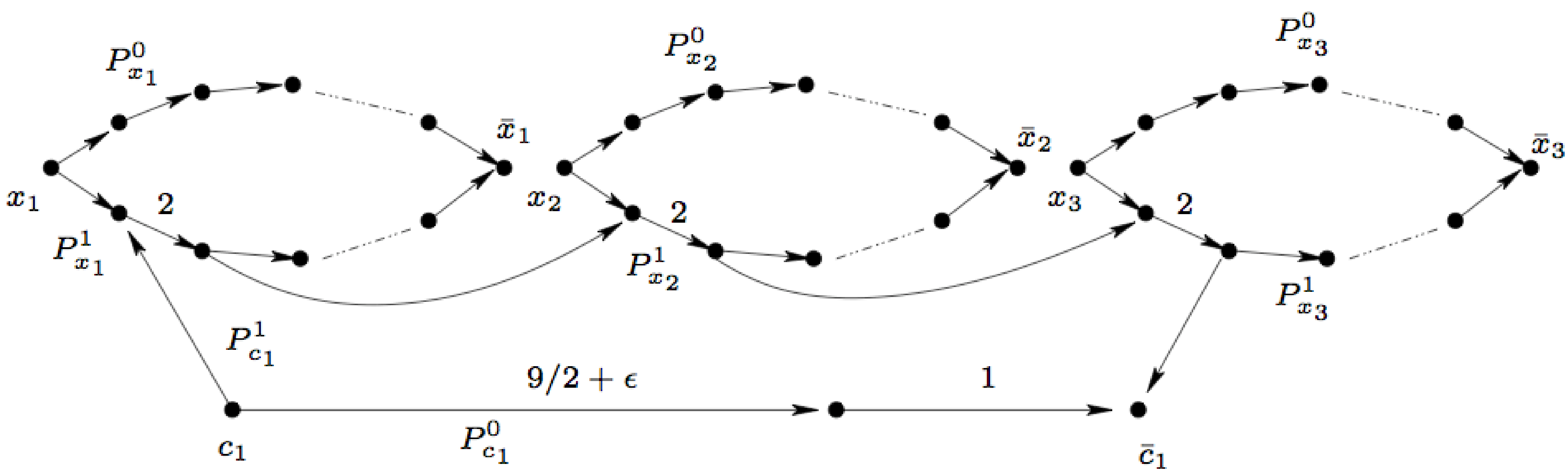

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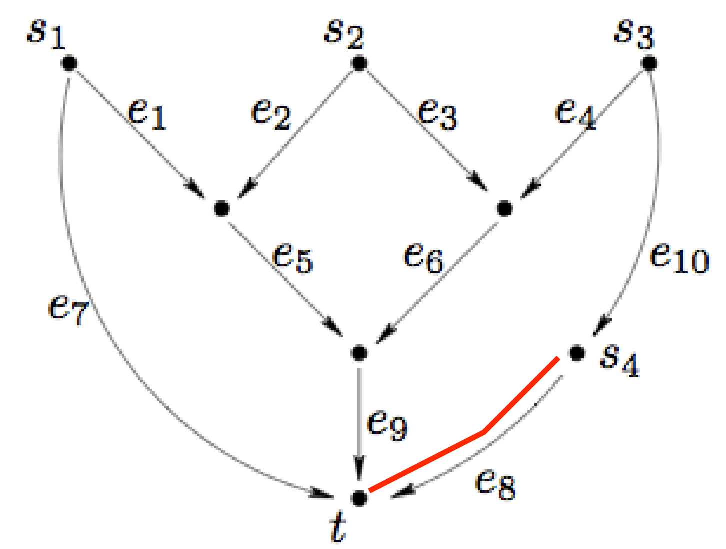


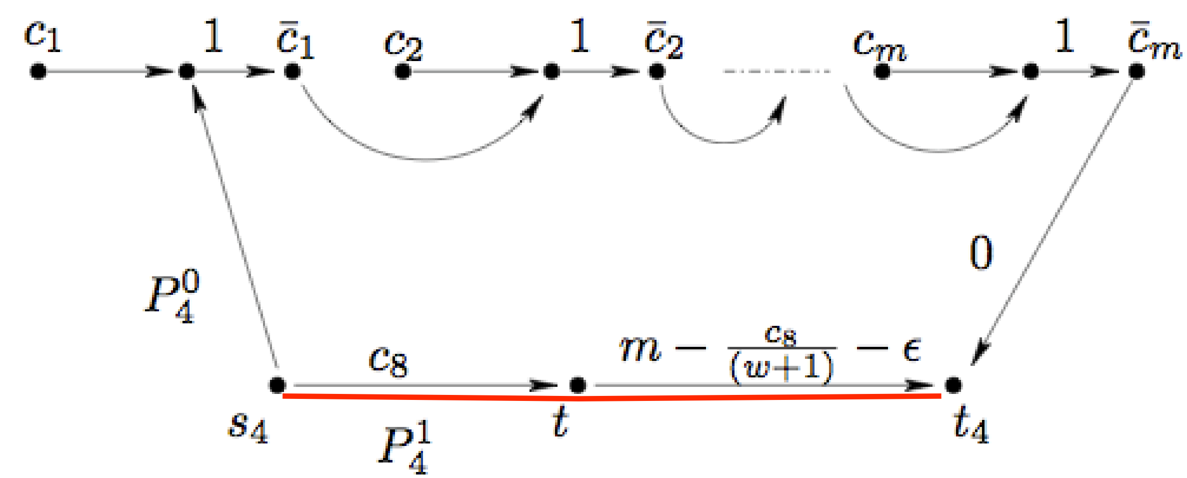


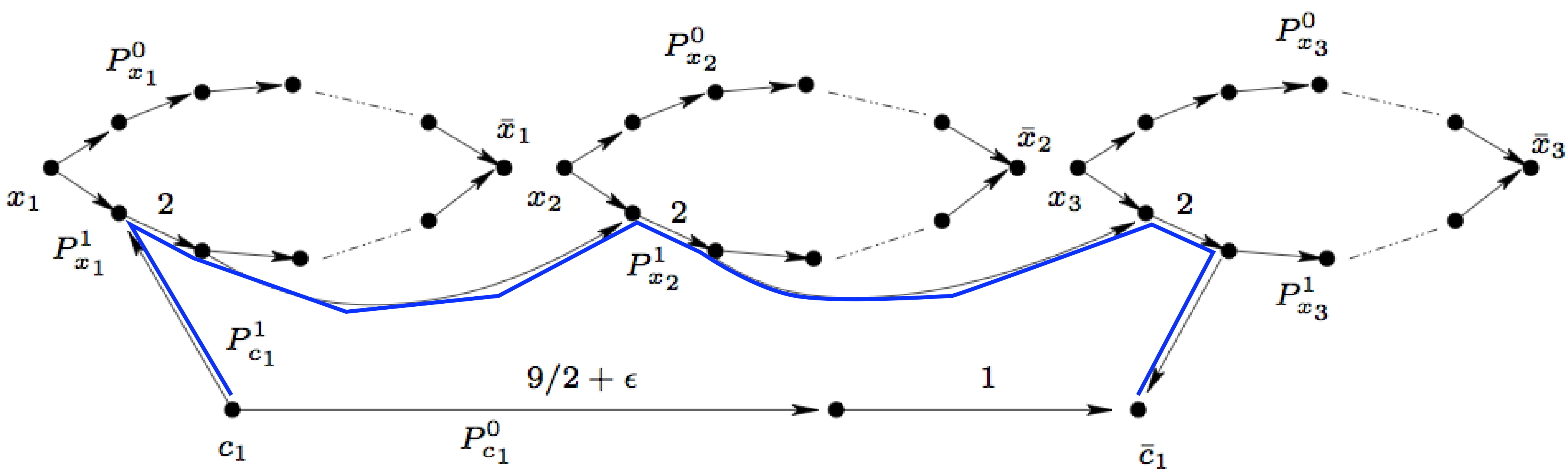

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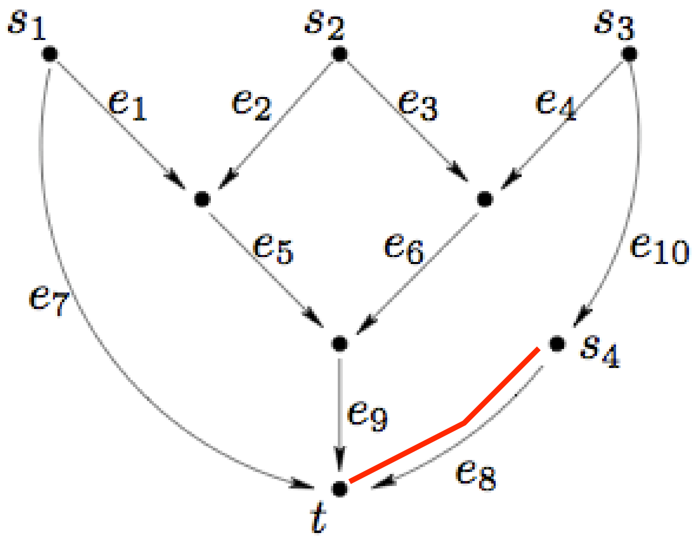
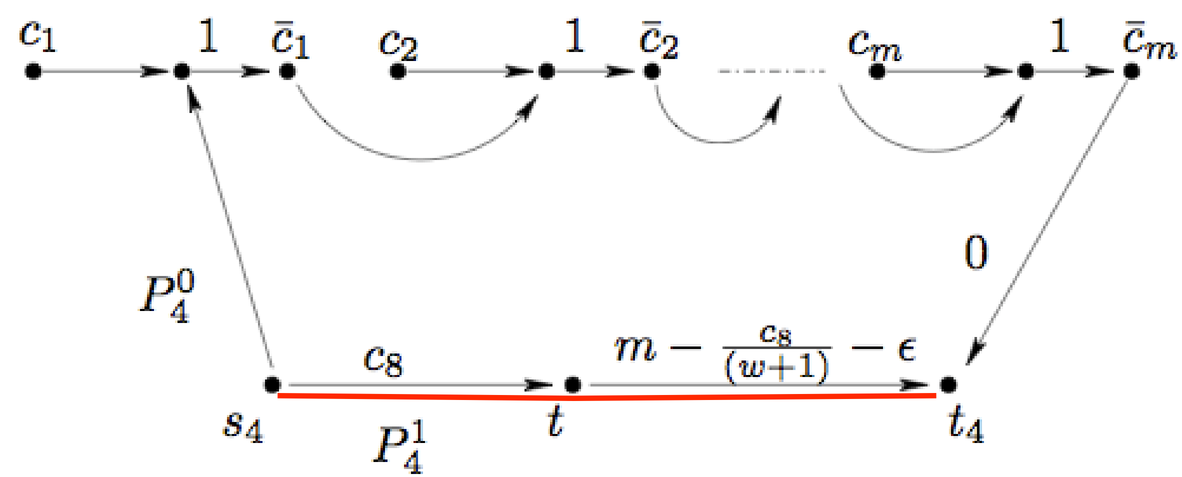
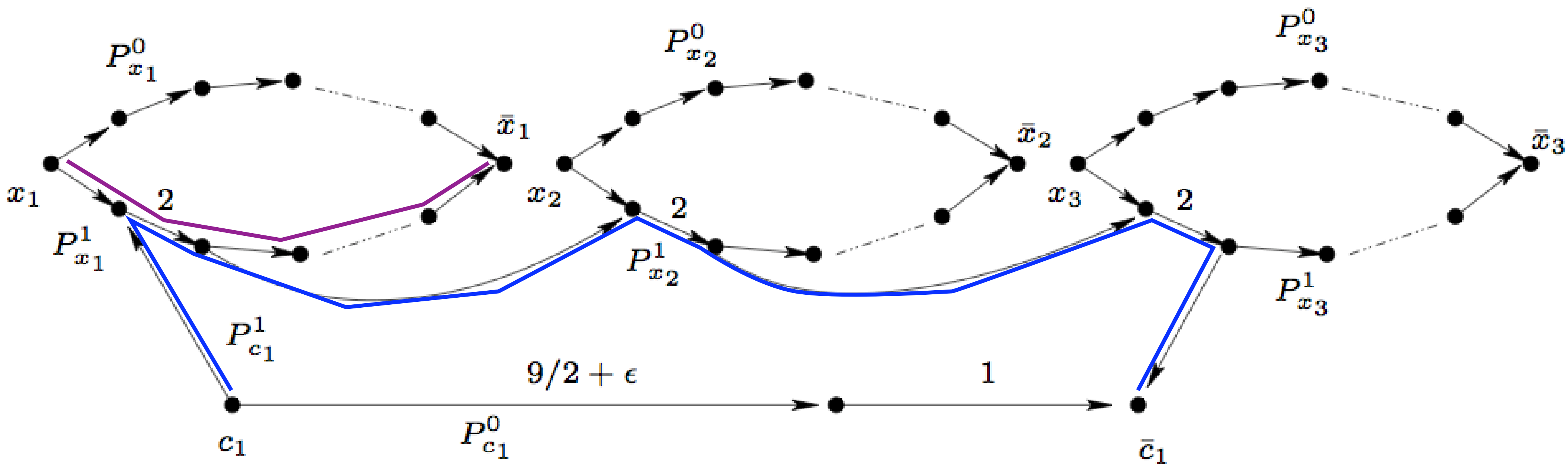
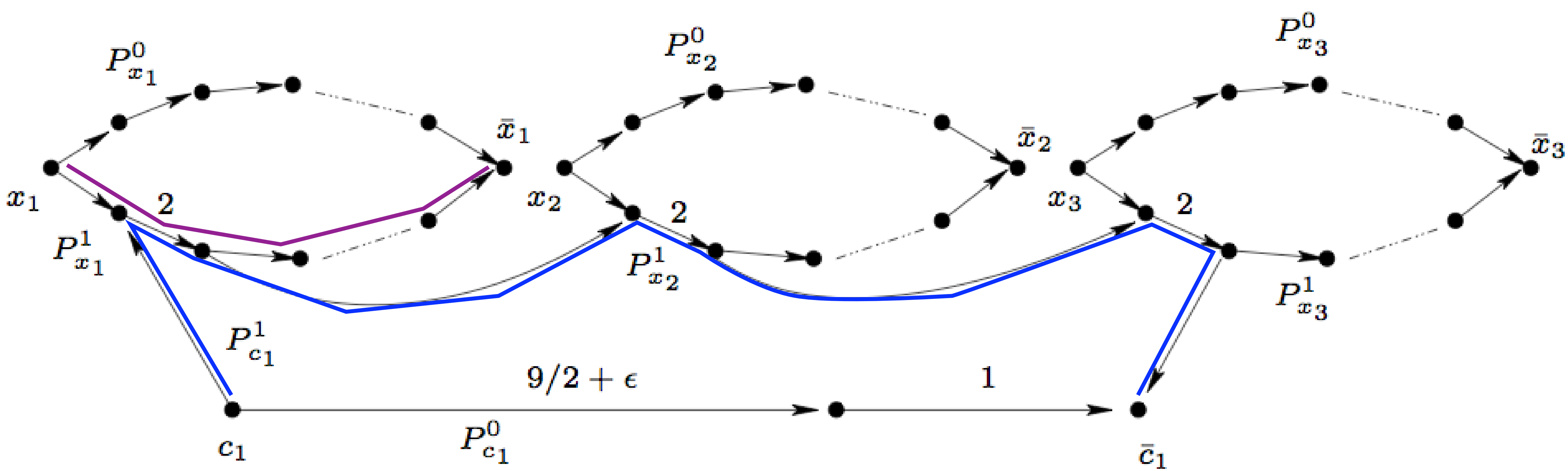


# Reduction Network

$$x = \begin{cases} 1 & \text{if } p_x \text{ uses 1-path,} \\ 0 & \text{if } p_x \text{ uses 0-path.} \end{cases}$$



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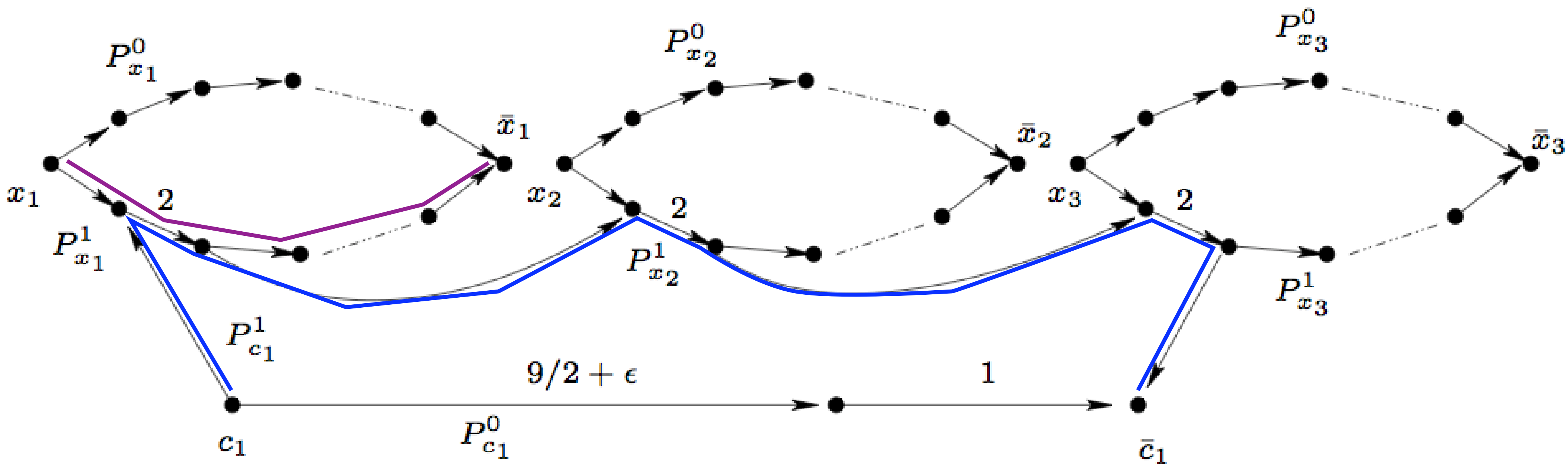
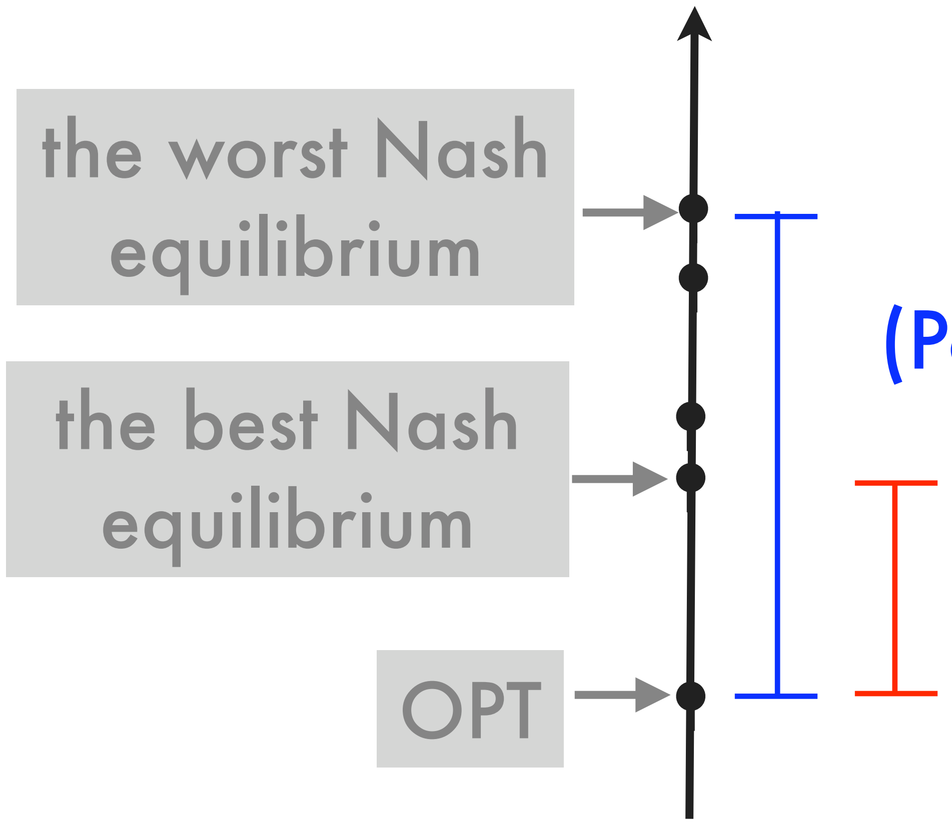
$$c = 1 \ \forall c$$



# Inefficiency of equilibria

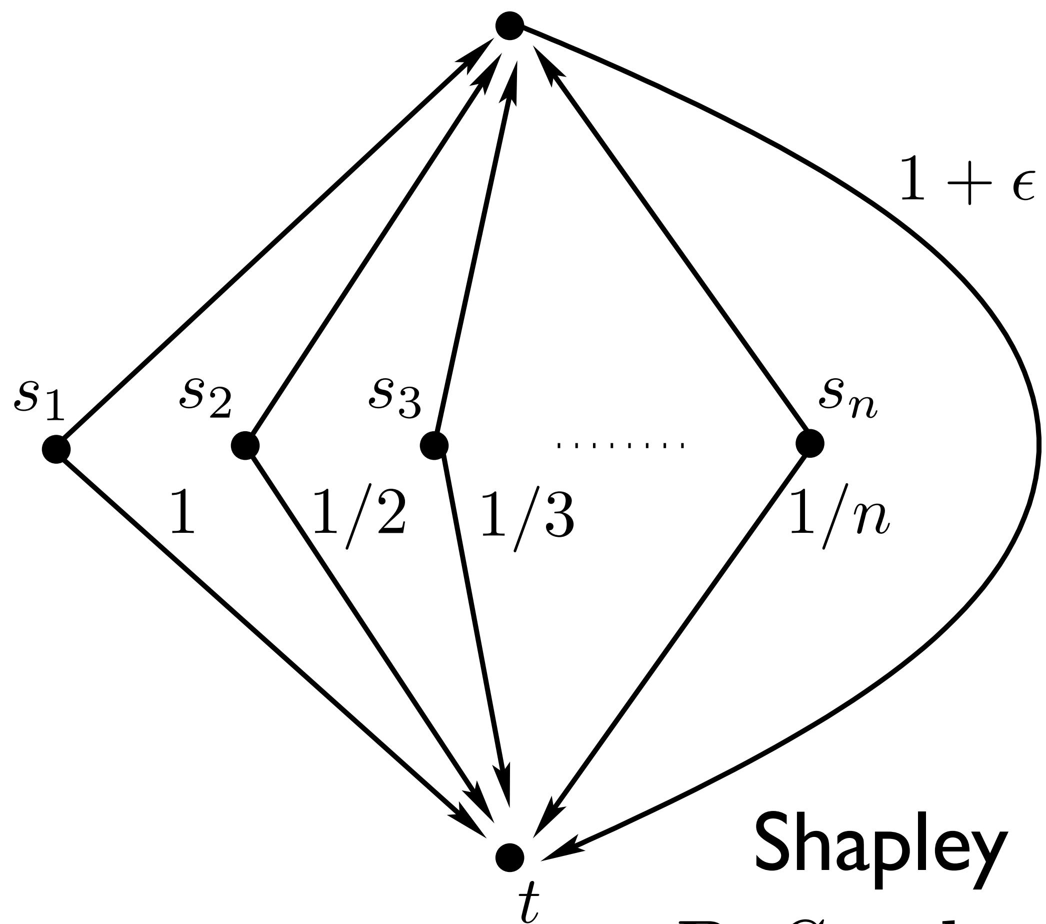
Good equilibria ?




price of anarchy (PoA) = worst NE/OPT

price of stability (PoS) = best NE/OPT

# Price of Stability




Shapley

$PoS = \log n$

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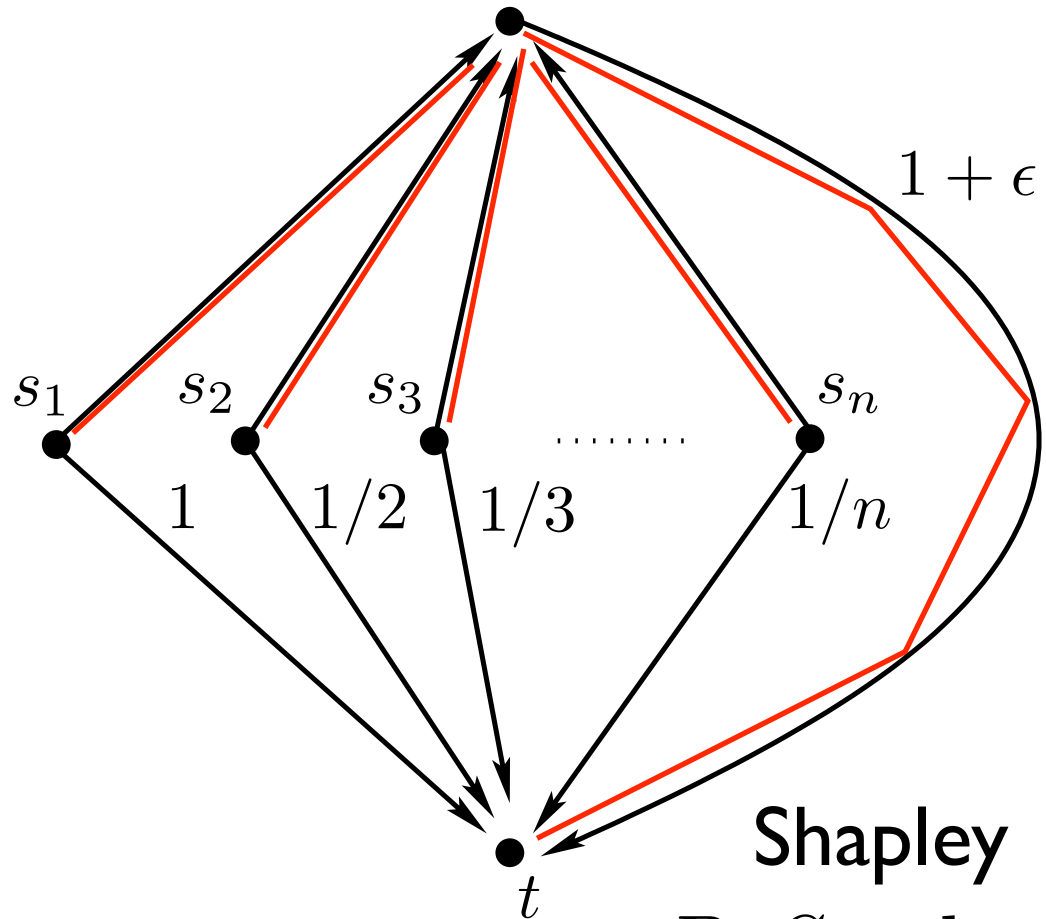


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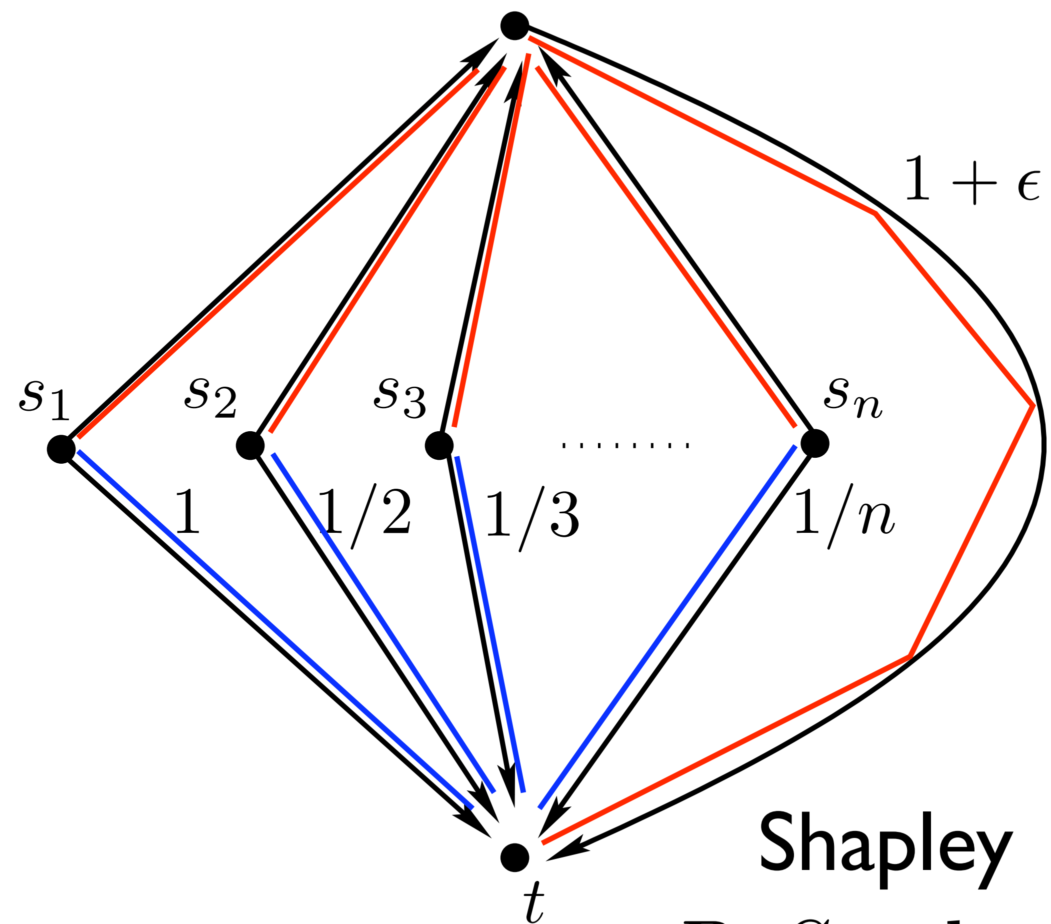


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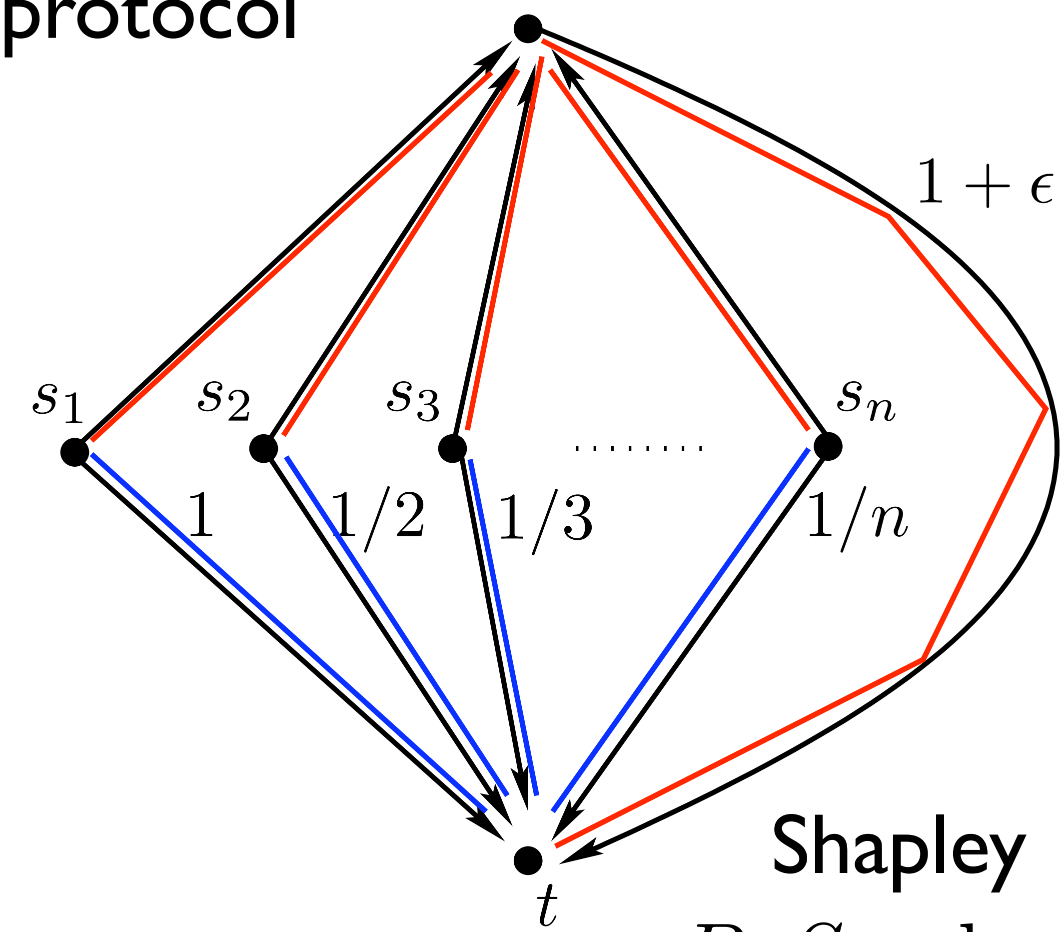
# Price of Stability

Why other cost-sharing protocol with small PoS ?

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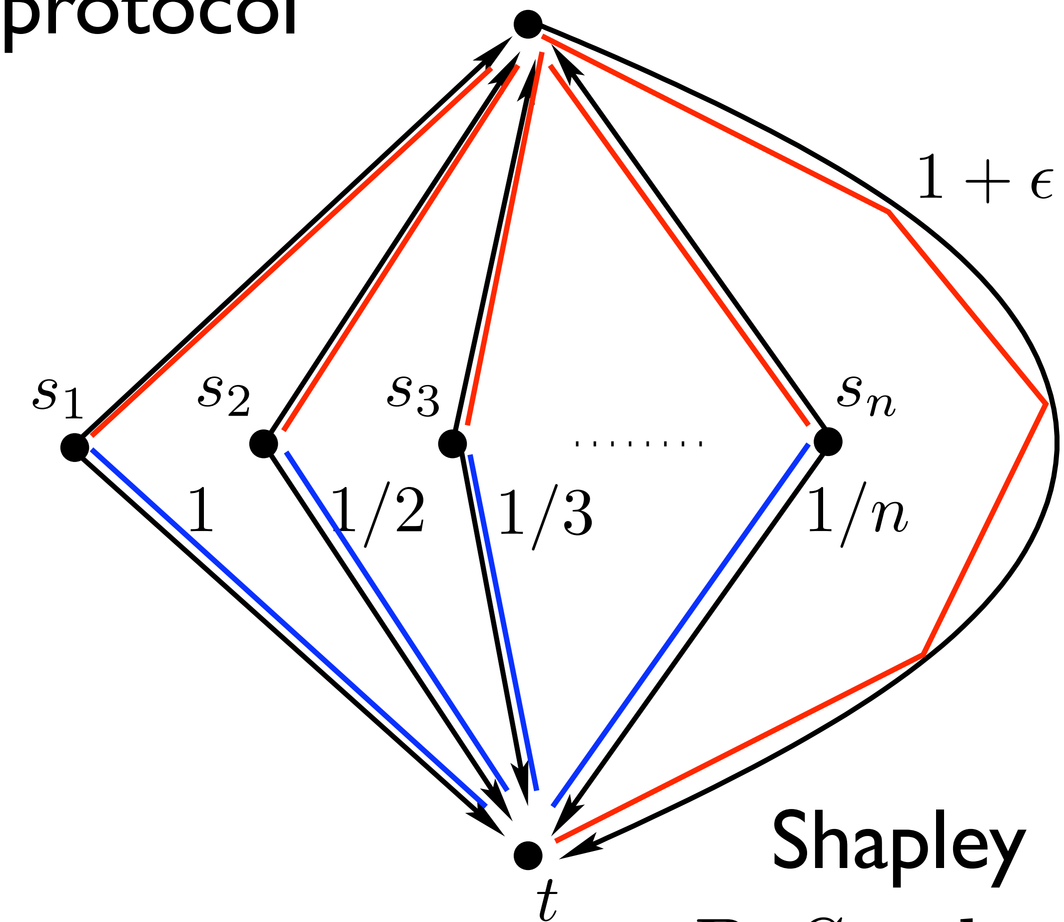
Why other cost-sharing protocol with small PoS ?

*$\epsilon$-admissible cost-sharing protocol:*

- *Budget balance*
- *Separability*
- *Stability*
- *$\epsilon$-fairness*

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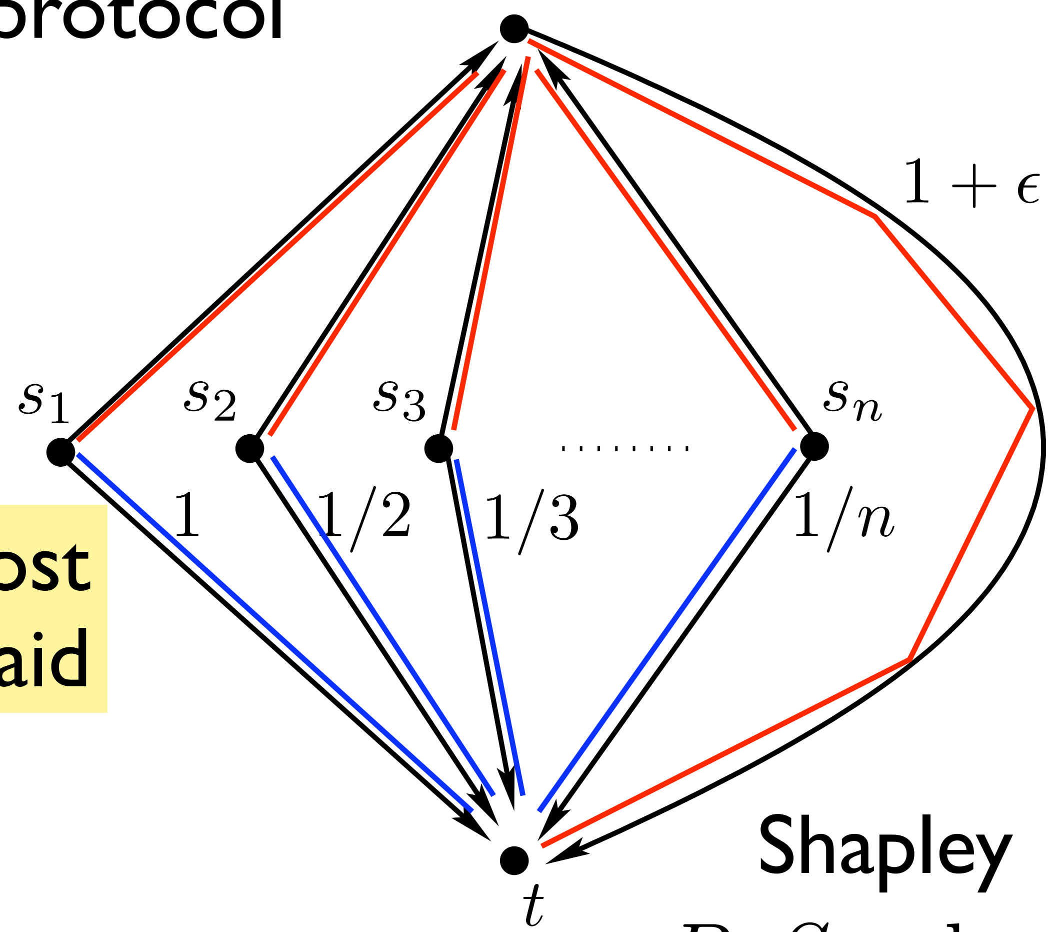
Edge's cost is fully paid

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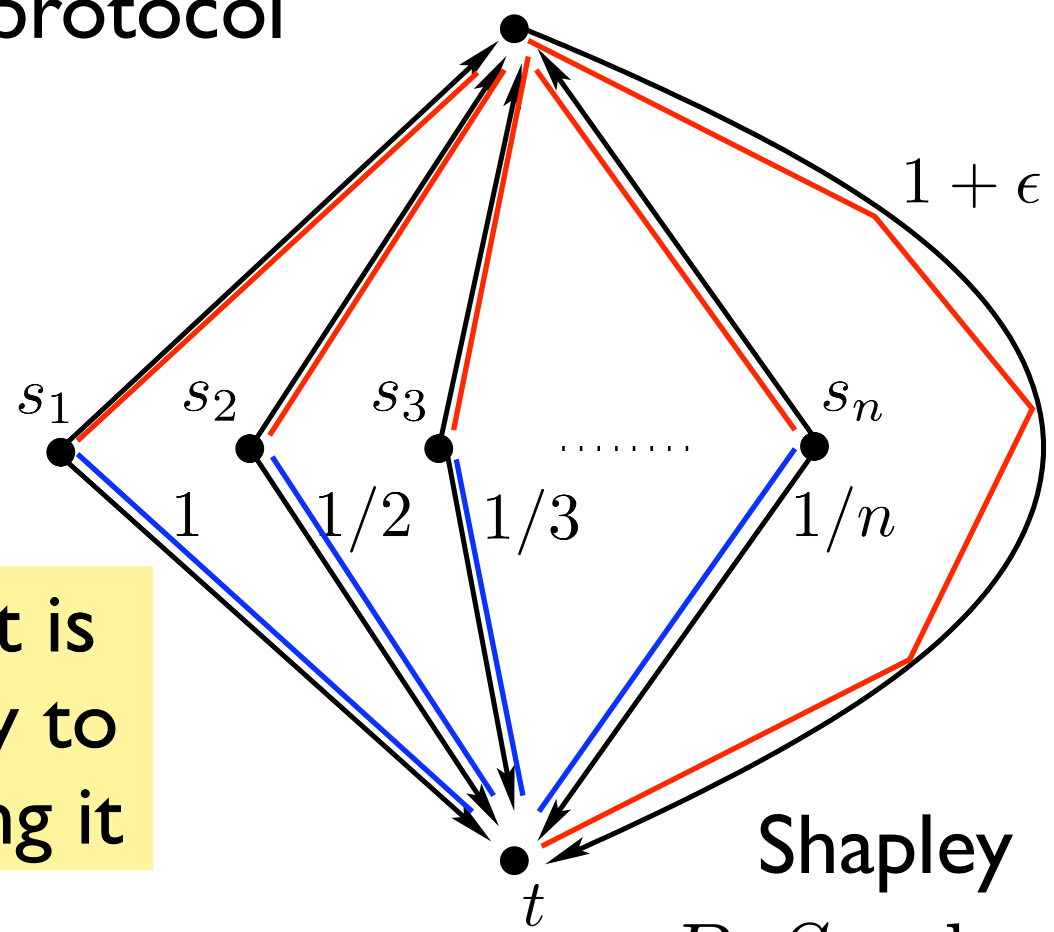
Edge's cost is shared only to players using it

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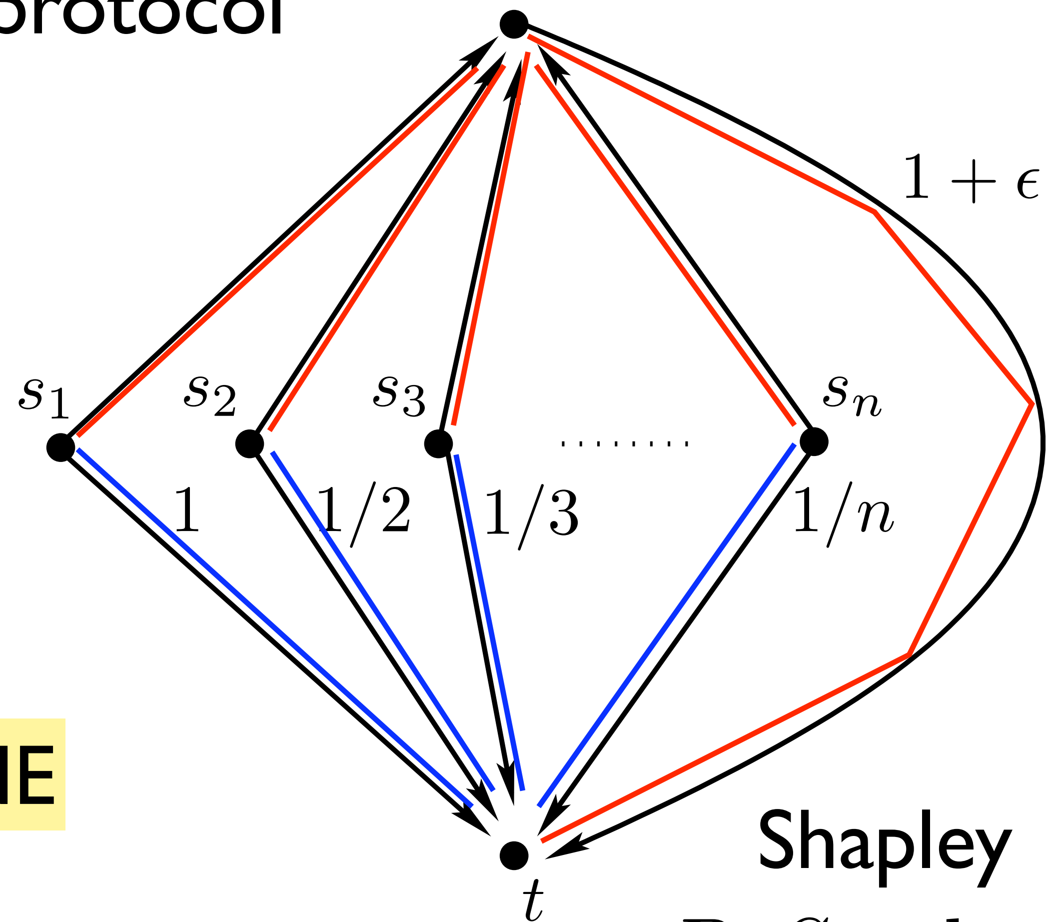
- *Separability*
- *Stability* There exists NE
- *$\epsilon$-fairness*

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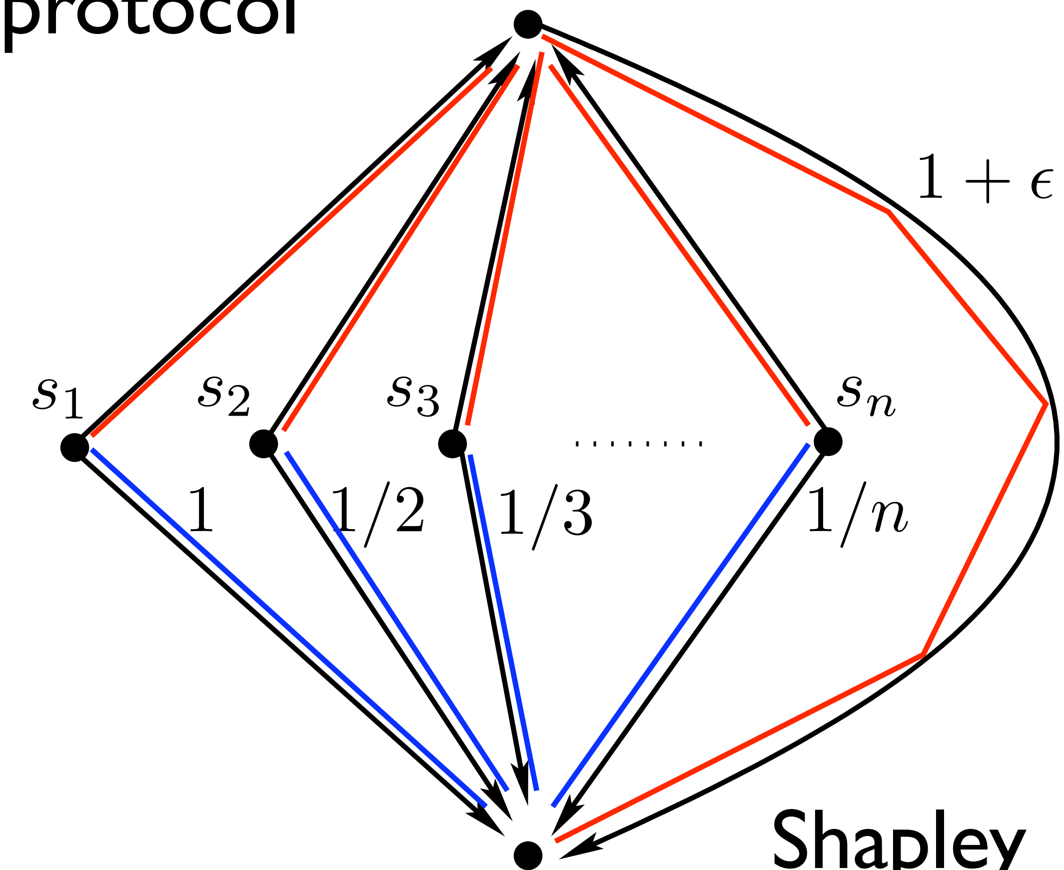


Shapley

$= \log n$

if there is more than two players using an edge then no one pays more than $(1-\epsilon)$ fraction of this edge's cost.

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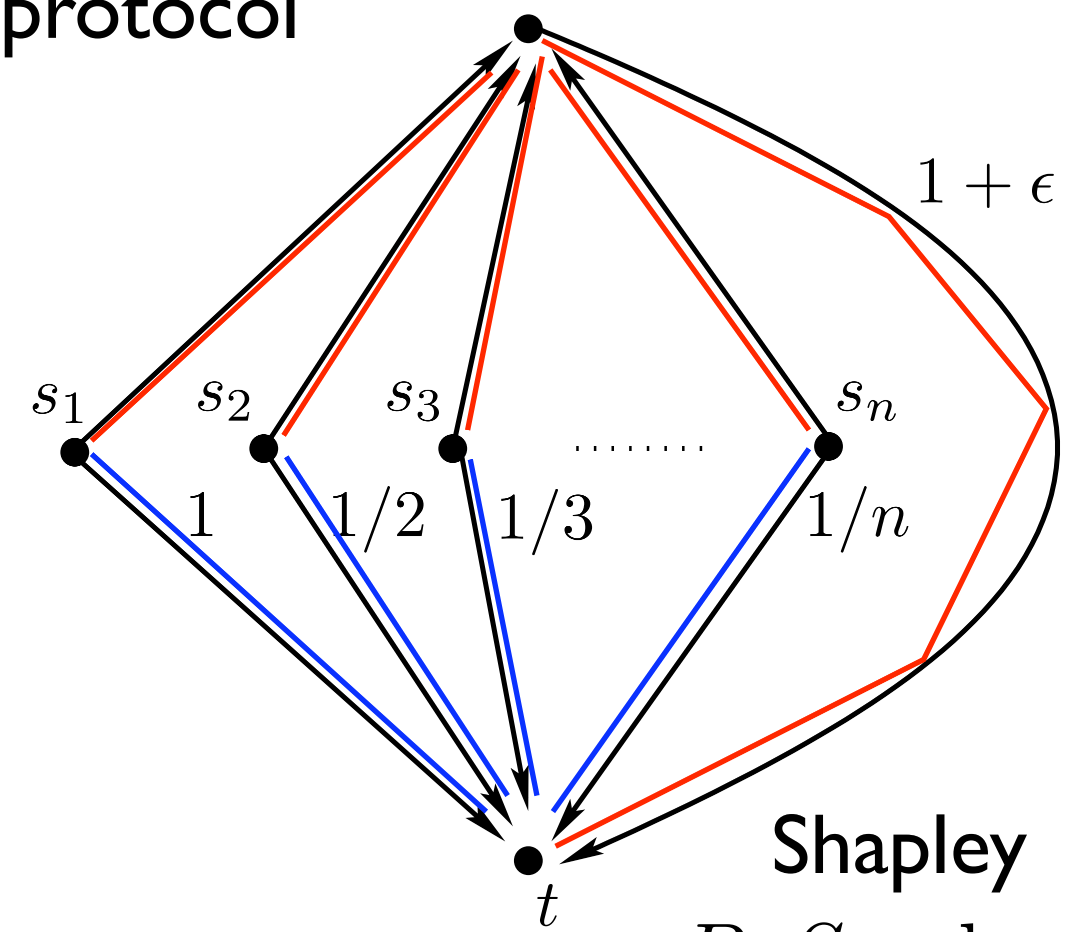


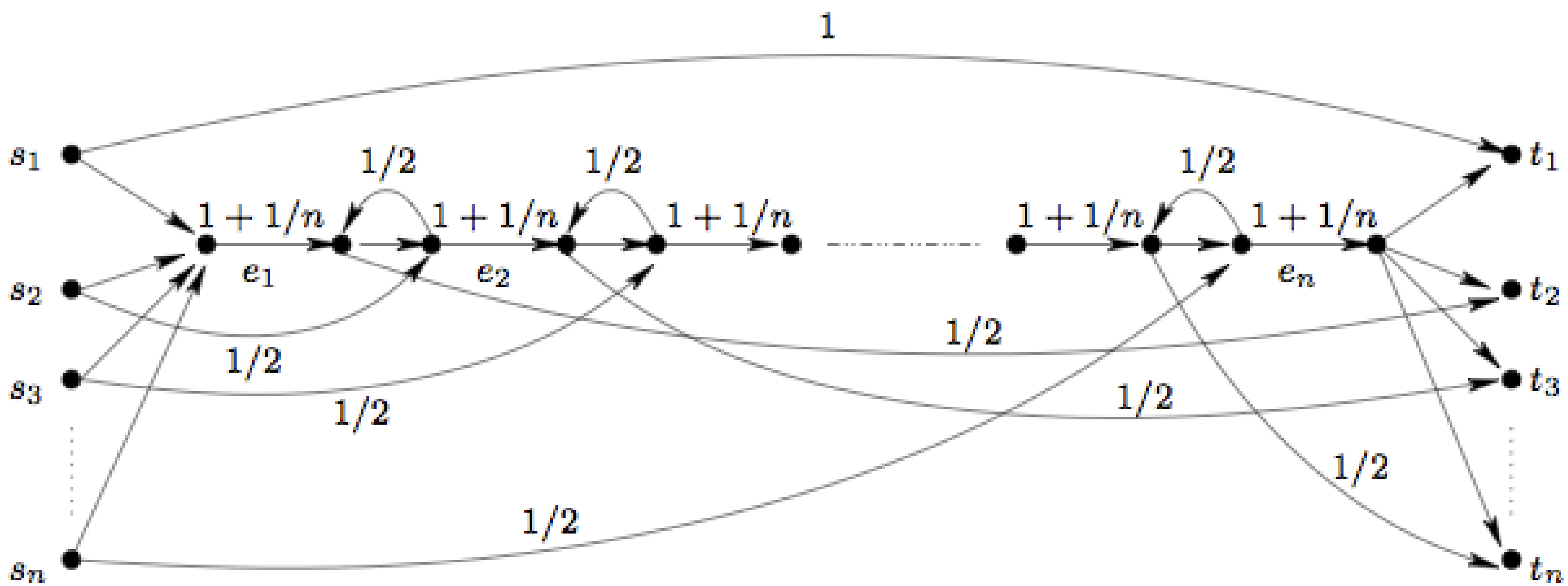

Shapley

$PoS = \log n$

Does there exist such a protocol with PoS = $1$ ?

# PoS in admissible cost-sharing

- *Lemma*: For any admissible cost-sharing protocol, PoS is at least 3/2.
- *Proof*:
  - OPT is the backbone path of cost $n+1$
  - In NE, no one uses entirely the backbone.
  - NE $\geq 3(n-1)/2$



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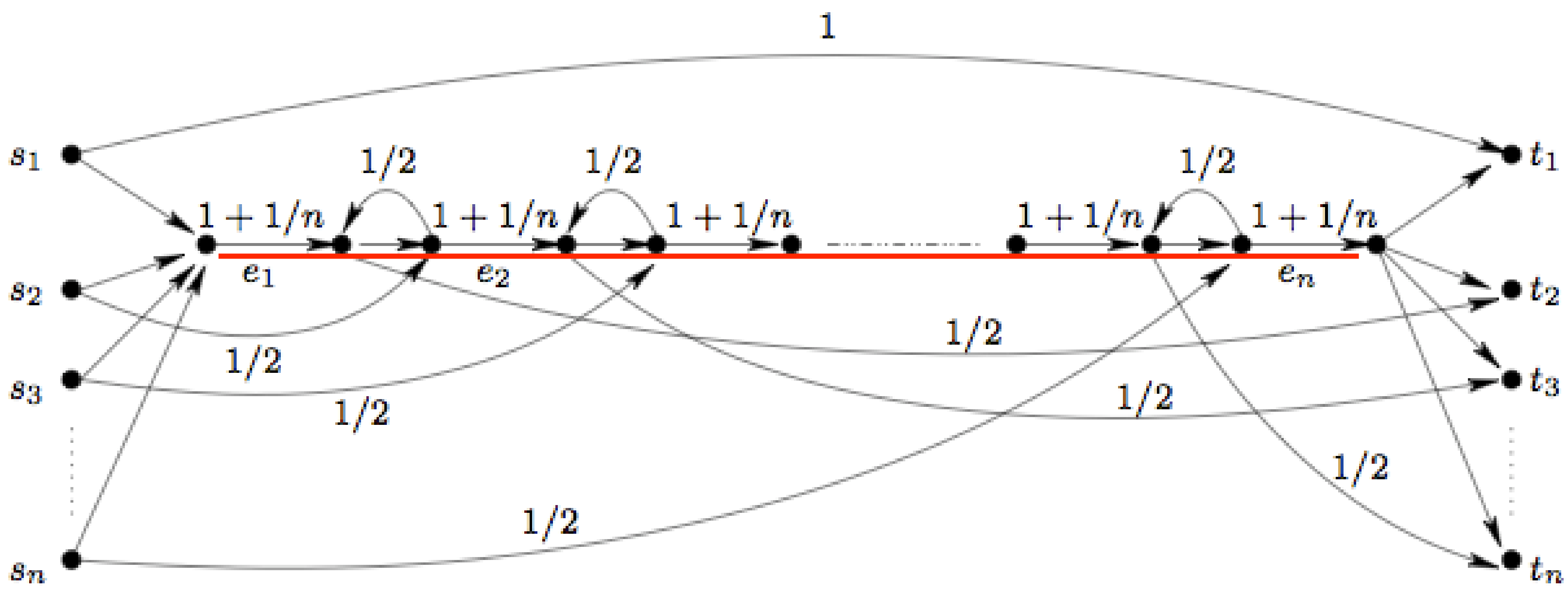

# NP-hardness

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- *Theorem*: Deciding if there exists an $\epsilon$-admissible cost-sharing protocol for a given network such that the PoS $\leq 3/2$ is NP-hard.

- *Proof*: Same technique to the previous proof.

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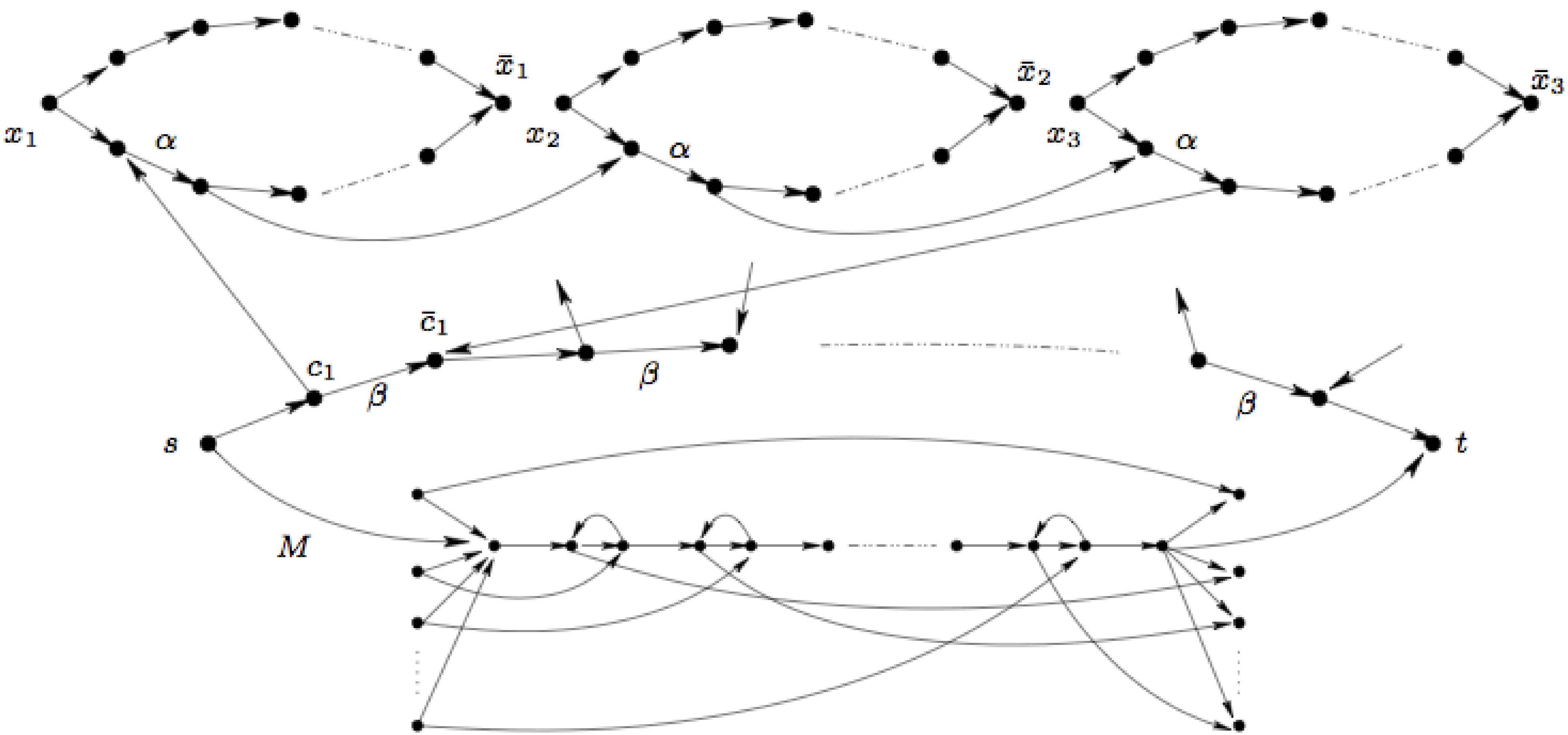
$c = (x_1 \vee x_2 \vee x_3)$ or $c = (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

# Reduction Network

$$2\alpha + (1-\epsilon)\alpha < \beta < 3\alpha$$

$$(m-1)\beta + (1-\epsilon)\alpha < M < M + 2 < m\beta$$

$$2(M + m\beta + nm\alpha) < k$$



# Conclusion and open question

☑ Present a technique in proving NP-hardness for problems about Nash equilibrium.

# Conclusion and open question

☑ Present a technique in proving NP-hardness for problems about Nash equilibrium.

☐ ? Designing cost-sharing protocol on (un)directed network with small PoS

# Conclusion and open question

*Thank you!*