Scheduling Games in the Dark

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Aarhus
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Outline

* Scheduling Games
    - Definition & Motivation
    - Summary of results
* Existence of pure Nash equilibrium
    - Potential argument
* Inefficiency of equilibria
    - Price of anarchy

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* Each machine specifies a policy how jobs assigned to the machine are to be scheduled (e.g., SPT, LPT, ...).

* The cost c_i of a job i is its completion time.

* The social cost is the makespan, i.e. $\max_j \ell_j = \max_i c_i$

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  - Local policy: depends only on the parameters of jobs assigned to it. $\sigma(i) = j \longrightarrow p_{ij'} \; \forall j'$

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  - MAKESPAN: if $\sigma(i) = j$

$$c_i = \ell_j$$

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- Local policies:
  - Inefficiency-based policy: greedily schedule jobs in increasing order of $\rho_i = p_{ij}/q_i$ where $q_i = \min_{j'} p_{ij'}$

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- Typically, a policy depends on the processing time of jobs assigned to the machine.

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  - Incomplete information games
  - Private information of jobs
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↙ existence Nash equilibrium

↘ small PoA

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  - A job i on machine j has an incentive to move to machine j' iff:

$$p_{ij} + \ell_j > 2p_{ij'} + \ell_{j'}$$

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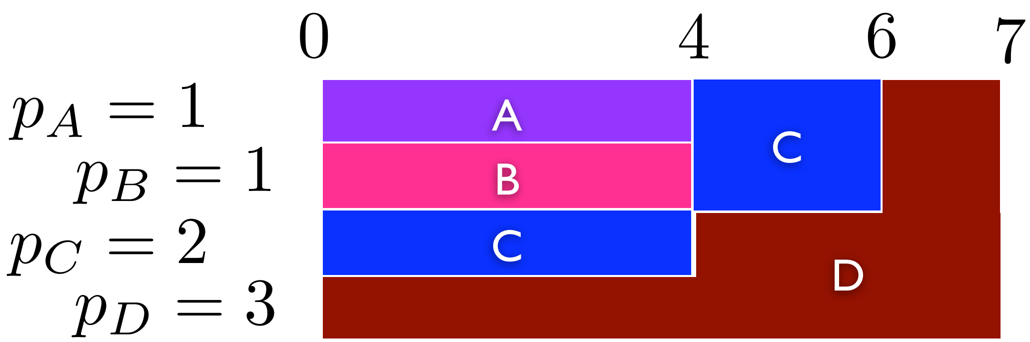

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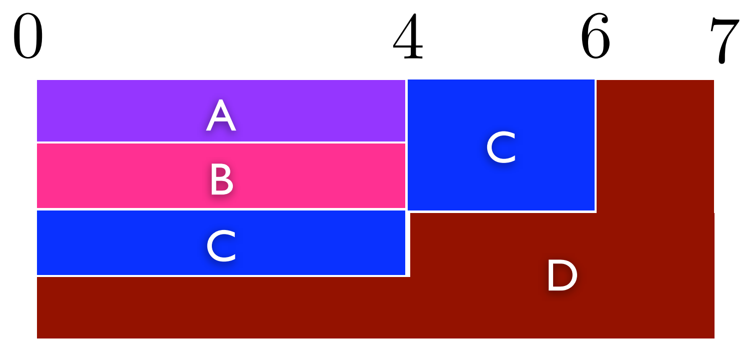
$p_A = 1$
$p_B = 1$
$p_C = 2$
$p_D = 3$



If there are k jobs on machine j s.t: $p_{1j} \leq \ldots \leq p_{kj}$

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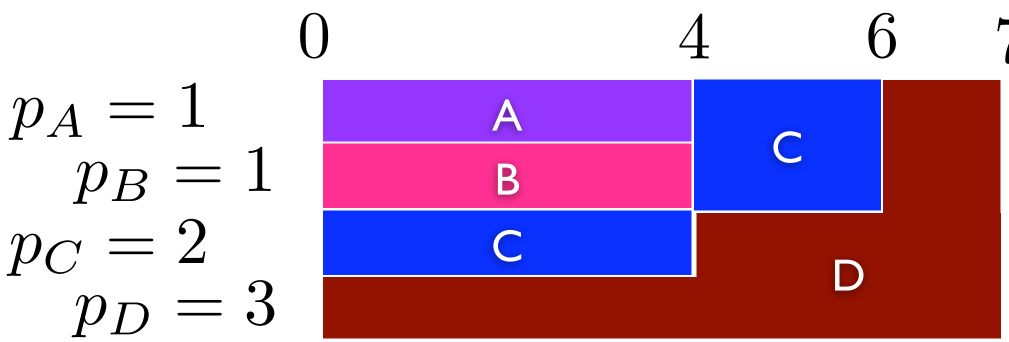

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sum ⟷ number

Models

* Def: A job i is balanced if $\max p_{ij} / \min p_{ij} \leq 2$

* Def of models:
  - Identical machines: $p_{ij} = p_i \ \forall j$ for some length p_i
  - Uniform machines: $p_{ij} = p_i / s_j$ for some speed s_j
  - Unrelated machines: p_{ij} arbitrary

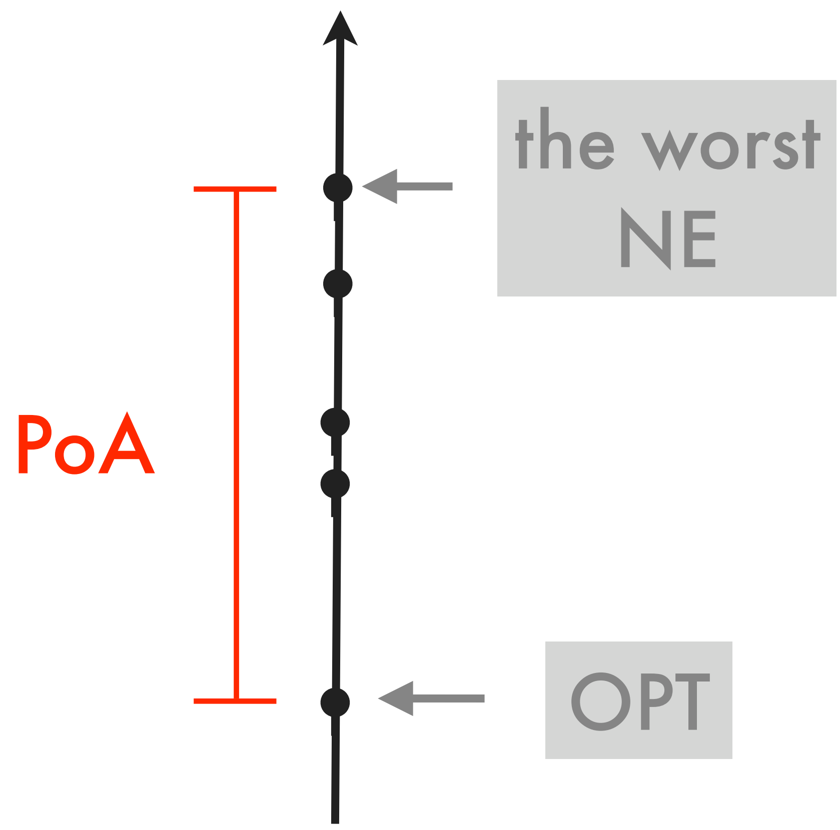
Existence of equilibrium

- Theorem:
  - The game under RANDOM policy is a potential game for 2 unrelated machines with balanced jobs but it is not for more than 3 machines. For uniform machines, balanced jobs, there always exists equilibrium.
  - The game under EQUI policy is a potential game.

Inefficiency

- Theorem: For unrelated machines, the PoA of policy EQUI is at most 2m – interestingly, that matches the best clairvoyant policy.

- PoA is not increased when processing times are unknown to the machines.



Existence of equilibrium

Standard definitions

* Def: A job is unhappy if it can decrease its cost by changing the strategy (other players' strategies are fixed)

* Def: a best response (best move) of a job is the strategy which minimizes the cost of the job (while other players' strategies are fixed)

* Def: Best-response dynamic is a process that let an arbitrary unhappy job make a best response.

RANDOM, uniform machines

- Theorem:
  - The game under RANDOM policy is a potential game for uniform machines with balanced jobs (balanced speeds).

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* Jobs have length $p_1 \leq p_2 \leq \ldots \leq p_n$
* Machines have speed $s_1 \geq s_2 \geq \ldots \geq s_m$

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* **Proof:** let i' be a new unhappy job.
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$$\ell_a + p_i/s_a > \ell_b + 2p_i/s_b$$

$$\ell_c + p_{i'}/s_c > (\ell_a - p_i/s_a) + 2p_{i'}/s_a$$

$$\ell_c + p_{i'}/s_c \leq \ell_b + 2p_{i'}/s_b$$

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Hence, $(s_a - s_b)(p_{i'} - p_i) > 0$

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  - The lemma follows.

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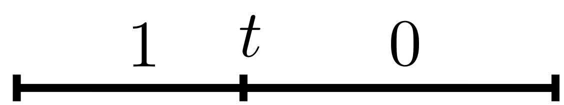
$$f_\sigma(i) = \begin{cases} 1 & \text{if } 1 \leq i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$



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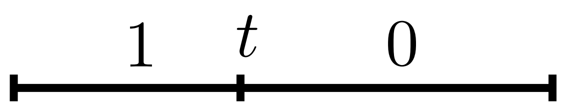

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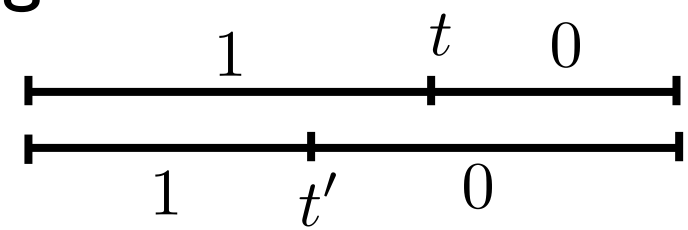

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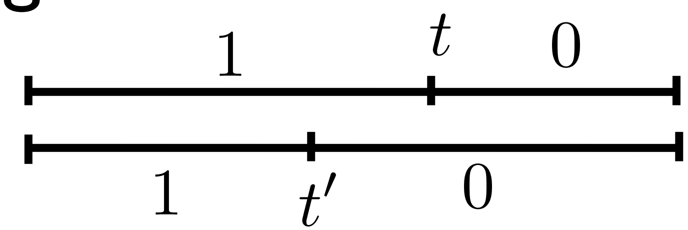

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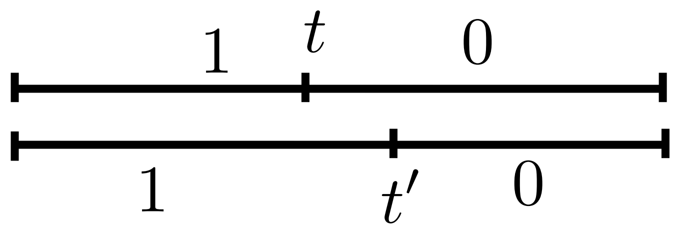

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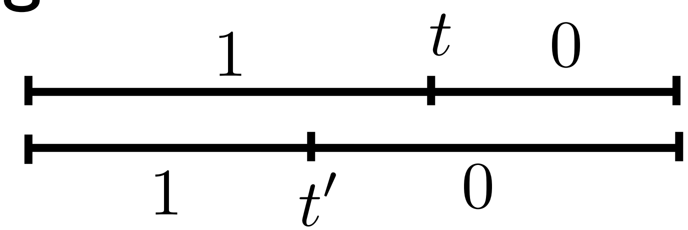

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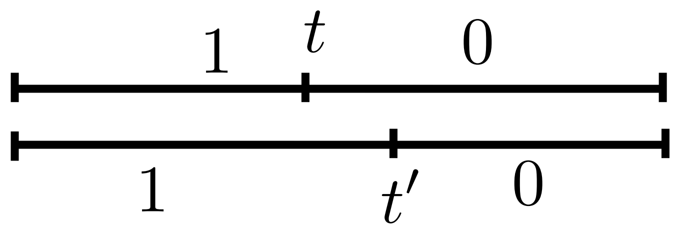

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- Proof:
  - Let $\sigma : \{1, \ldots, n\} \to \{ 1, 2\}$ be the current profile.
  - The following potential decreases strictly

$$\Phi = |\ell_1 - \ell_2| + 3 \sum_i \max\{p_{i\sigma(i)} - p_{i\overline{\sigma(i)}}, 0\}$$

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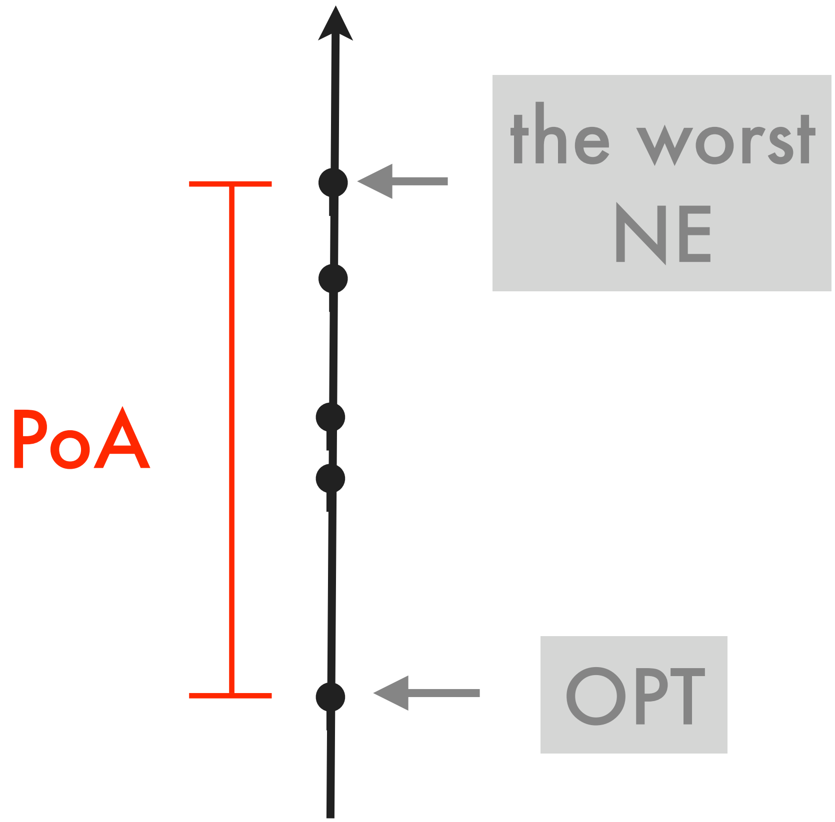
Inefficiency of equilibria

Inefficiency

* **Theorem:** For unrelated machines, the PoA of policy EQUI is at most 2m.

* The knowledge about jobs' characteristics is not necessarily needed.



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- Proof: $$q_i := \min_j p_{ij}$$ $$\sum_{i=1}^{n} q_i \leq m \cdot OPT$$ $$Q(i) := \arg\min_j p_{ij}$$
  - Rename jobs such that $q_1 \leq q_2 \leq \ldots \leq q_n$

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- Proof: $c_1 \leq nq_1$ = the worst cost on $Q(1)$

$c_2 \leq q_1 + (n-1)q_2$ = the worst cost on $Q(2)$

Proof (sketch)

- By monotonicity of $(q_i)_{i=1}^n$

$$\text{makespan} = \max_i c_i$$
$$\leq \max_i (2q_1 + \ldots 2q_i + (n-i+1)q_i)$$
$$\leq 2\sum_i q_i$$
$$\leq 2m \cdot OPT$$

$$PoA \leq 2m$$

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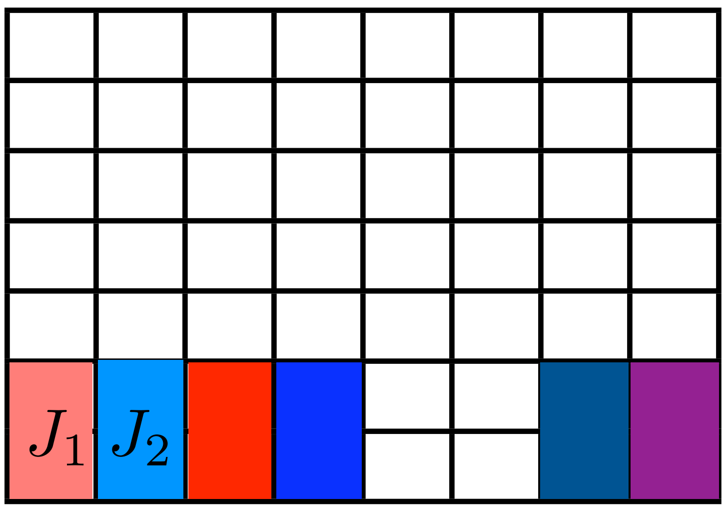

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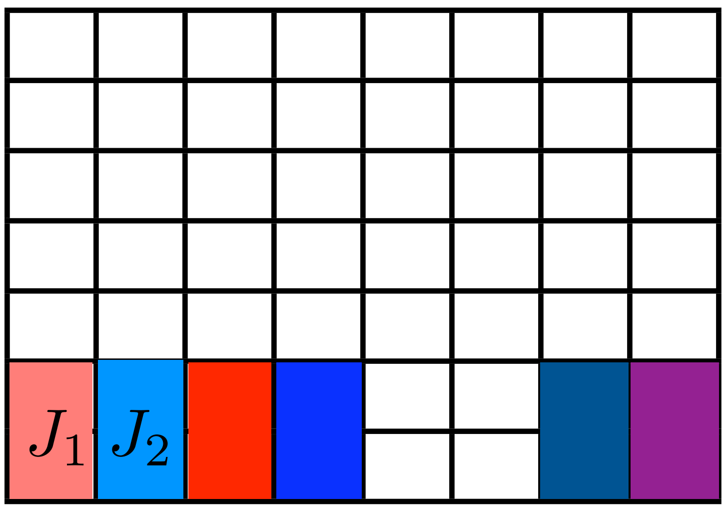
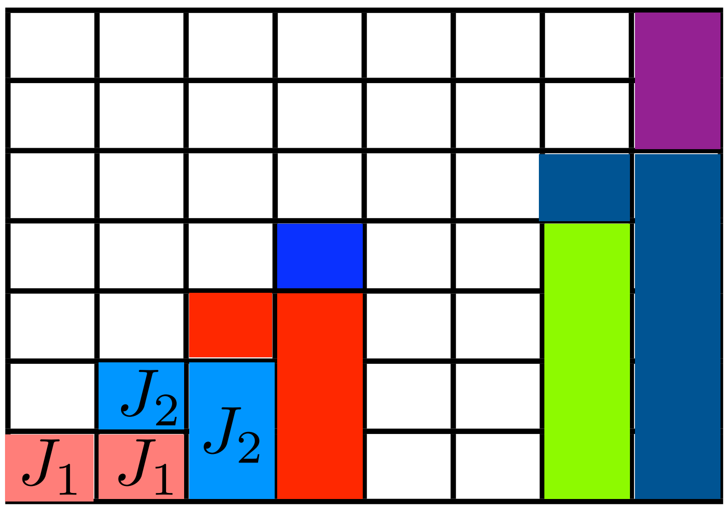


NE



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☐ Designing local policy with PoA = $o(\log m)$