On (Group) Strategy-proof Mechanisms without Payment for Facility Location Games

Nguyen Kim Thang

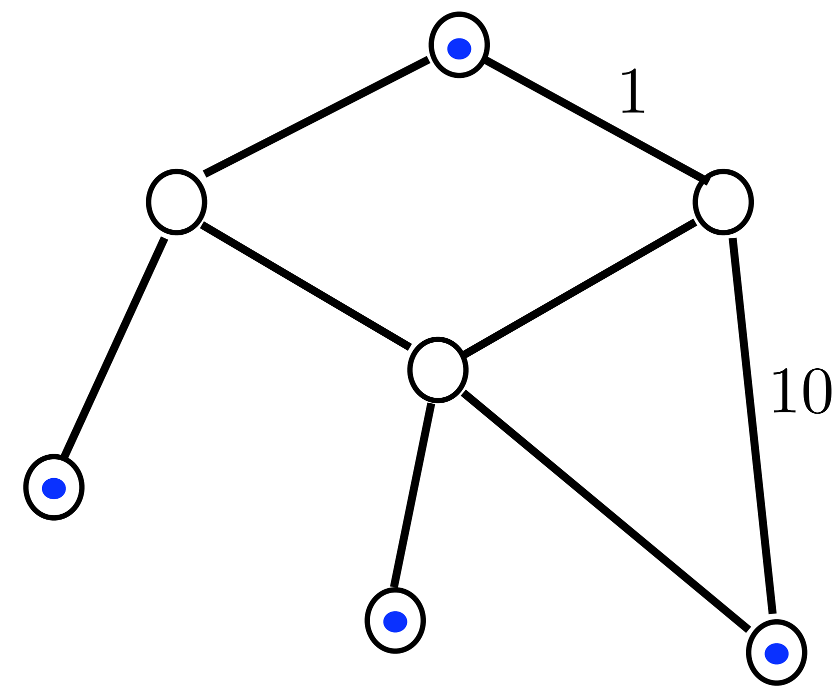
WINE'10

Outline

- Facility Games
  - Definitions
  - Results
- ☑ Our main results
  - High level ideas.
- Conclusion & Further directions

Facility Games

- A network is represented by a graph $G(V, E)$
- $d(u, v) =$ minimum-length path.
- n agents, agent i has location $x_i \in V$



- A det mechanism $f : V^n \to V$

$$\mathbf{x} = \langle x_1, \ldots, x_n \rangle \mapsto F$$

Agent's cost: $d(x_i, F)$

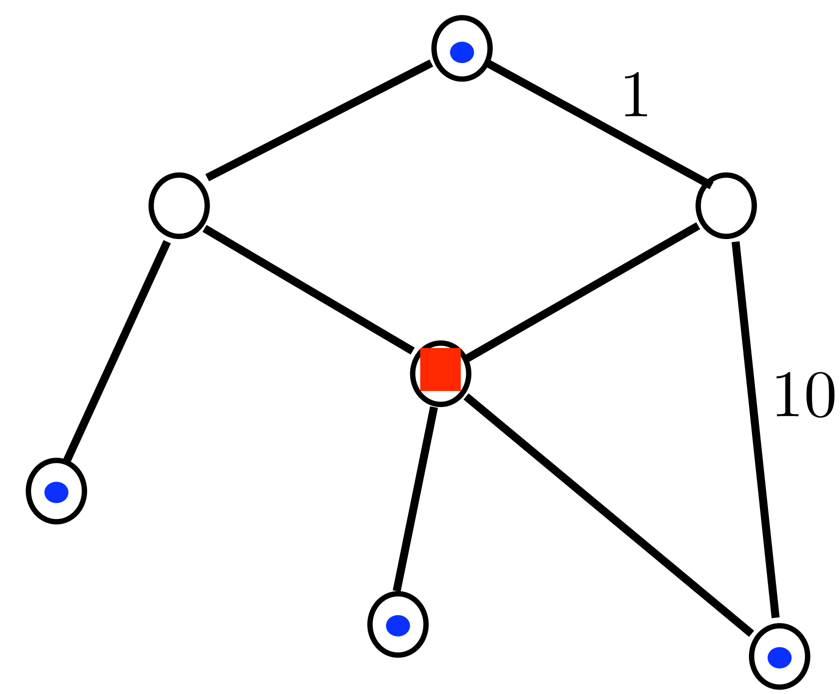
- A random mechanism $f : V^n \to \Delta(V)$

$$\mathbf{x} = \langle x_1, \ldots, x_n \rangle \mapsto P$$

Agent's cost: $\mathbb{E}_{F \sim P}[d(x_i, F)]$

Facility Games

- A network is represented by a graph $G(V,E)$
- $d(u,v) =$ minimum-length path.
- n agents, agent i has location $x_i \in V$



- A det mechanism $f : V^n \to V$

$$\mathbf{x} = \langle x_1, \ldots, x_n \rangle \mapsto F$$

Agent's cost: $d(x_i, F)$

- A random mechanism $f : V^n \to \Delta(V)$

$$\mathbf{x} = \langle x_1, \ldots, x_n \rangle \mapsto P$$

Agent's cost: $\mathbb{E}_{F \sim P}[d(x_i, F)]$

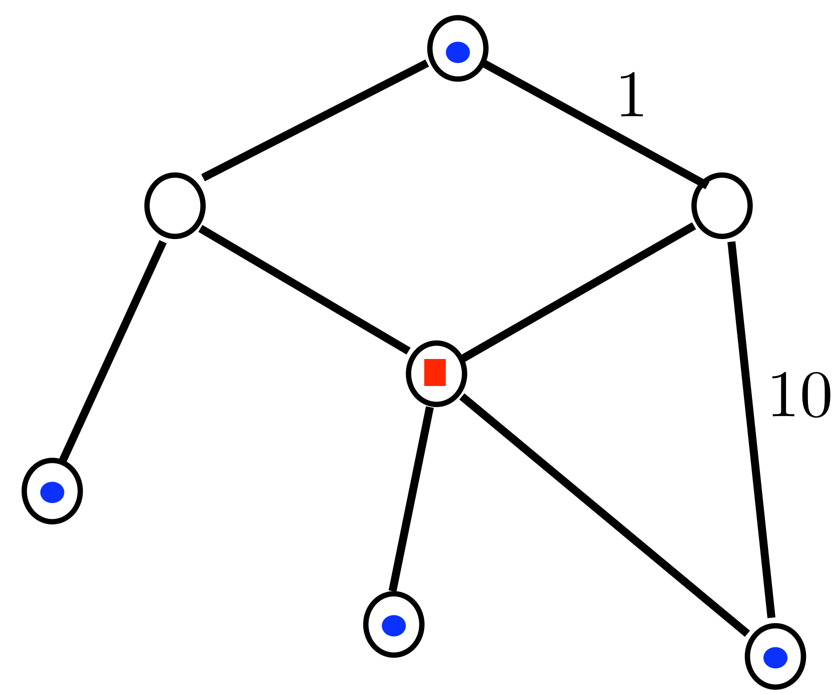
Facility Games

- A mechanism is strategy-proof if $\forall i$

$$cost(x_i, f(x'_i, \mathbf{x}_{-i})) \geq cost(x_i, f(\mathbf{x}))$$

- A mechanism is group strategy-proof if for all $S \subset N$, there exists $i \in S$

$$cost(x_i, f(x'_S, \mathbf{x}_{-S})) > cost(x_i, f(\mathbf{x}))$$



- Social objective functions: $\sum_{i \in N} cost(x_i, F)$

- A mechanism f is α-approximation if

$$cost(f) \leq \alpha \cdot OPT$$

Results

	Line graphs	General graphs
Det	$1\ GSP$	$n\ GSP$ (upper bound) $(n-1)\ SP$ (lower bound)
Ran	$1\ GSP$	

Results

	Line graphs	General graphs
Det	$1\ GSP$	$n\ GSP$ (upper bound) $(n-1)\ SP$ (lower bound)
Ran	$1\ GSP$	$(2-2/n)\ SP, n\ GSP$ (u.b)

Results

	Line graphs	General graphs
Det	$1\ GSP$	$n\ GSP$ (upper bound) $(n-1)\ SP$ (lower bound)
Ran	$1\ GSP$	$(2-2/n)\ SP, n\ GSP$ (u.b) $\left(2-\frac{4}{n^{1/3}}\right)\ SP, \Omega(n^{1-\epsilon})\ GSP$

Results

	Line graphs	General graphs
Det	$1 \; GSP$	$n \; GSP$ (upper bound) $(n-1) \; SP$ (lower bound)
Ran	$1 \; GSP$	$(2 - 2/n) \; SP, n \; GSP$ (u.b) $\left(2 - \frac{4}{n^{1/3}}\right) \; SP, \Omega(n^{1-\epsilon}) \; GSP$

Designing (G)SP mechanism without money is expensive

Framework of lower bound

Fix a graph

- An instance differs from the previous one in some agents' locations
- Connect instances using strategy-proofness.

GSP mechanisms

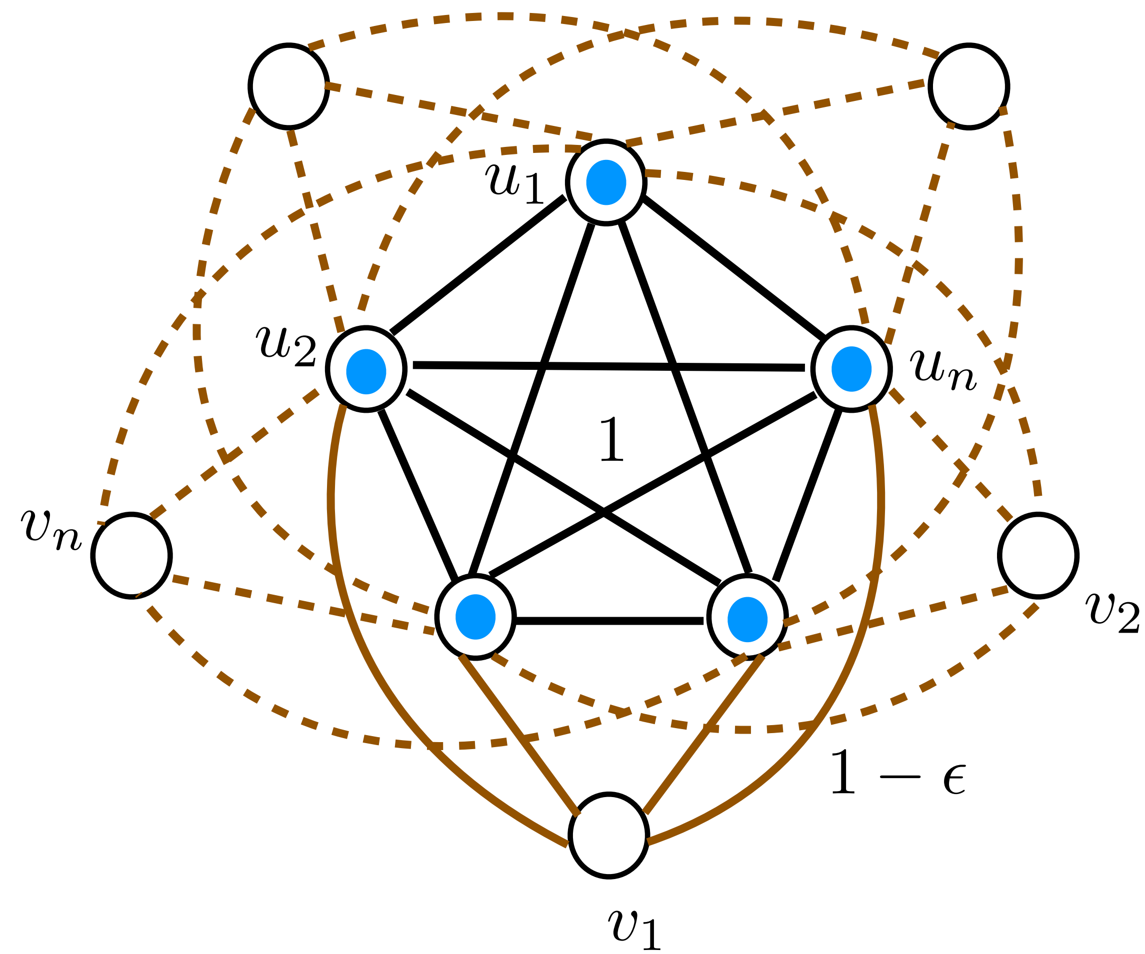
- Dictatorship: open the facility at location of some fixed agent.
  - ☑ GSP and n-approximation.
  - ☑ Deterministic lower bound by theorem of Schummer and Vohra. (Any SP mechanism on a cycle graph is dictatorship.)

GSP mechanisms

- Dictatorship: open the facility at location of some fixed agent.
  - ☑ GSP and n-approximation.
  - ☑ Deterministic lower bound by theorem of Schummer and Vohra. (Any SP mechanism on a cycle graph is dictatorship.)

☑ Thm: no randomized GSP mechanism is better than $n^{1-3\epsilon}/3$ - approximation.

Deterministic GSP mechanisms

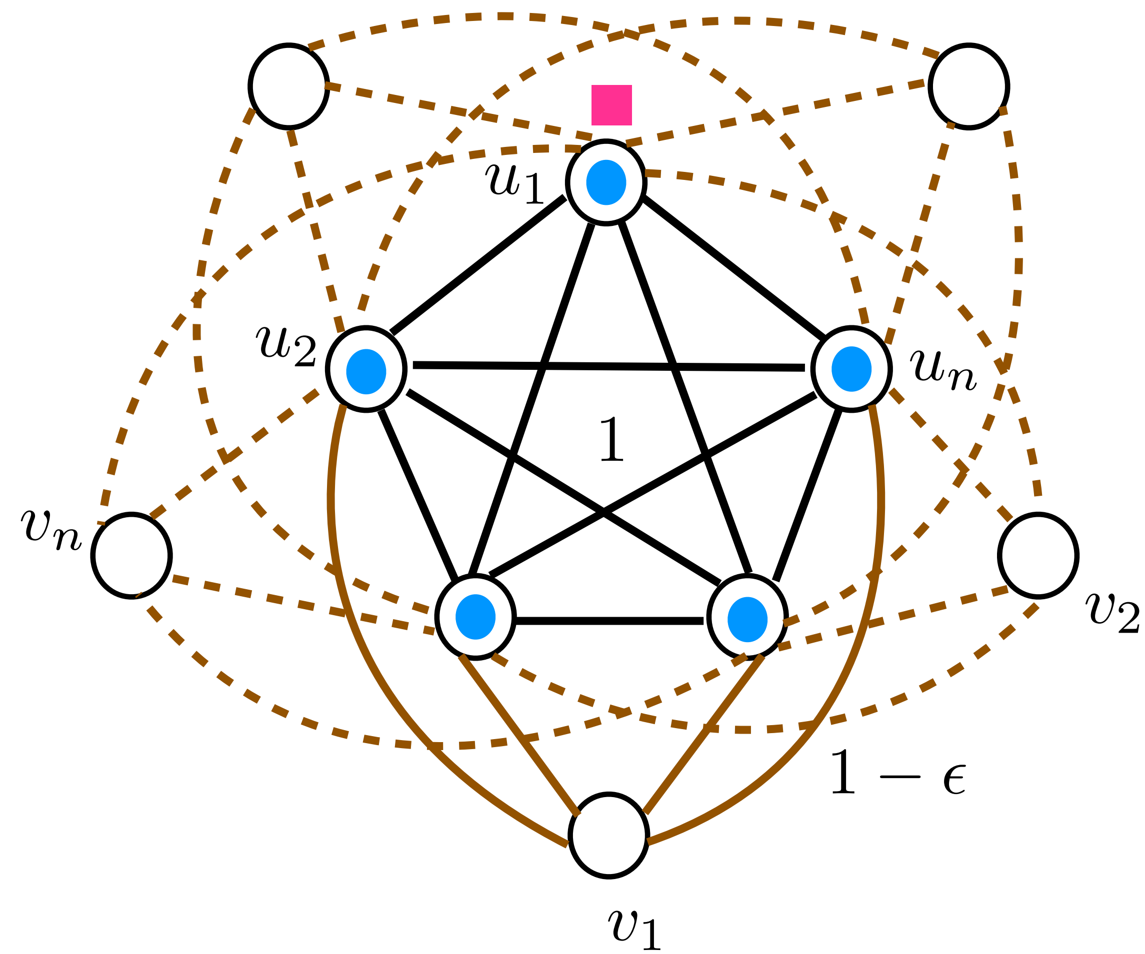


Deterministic GSP mechanisms

- Case 1: facility is opened in U

  All agents but the first one move to v_1. The facility is not opened at v_1

$$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$

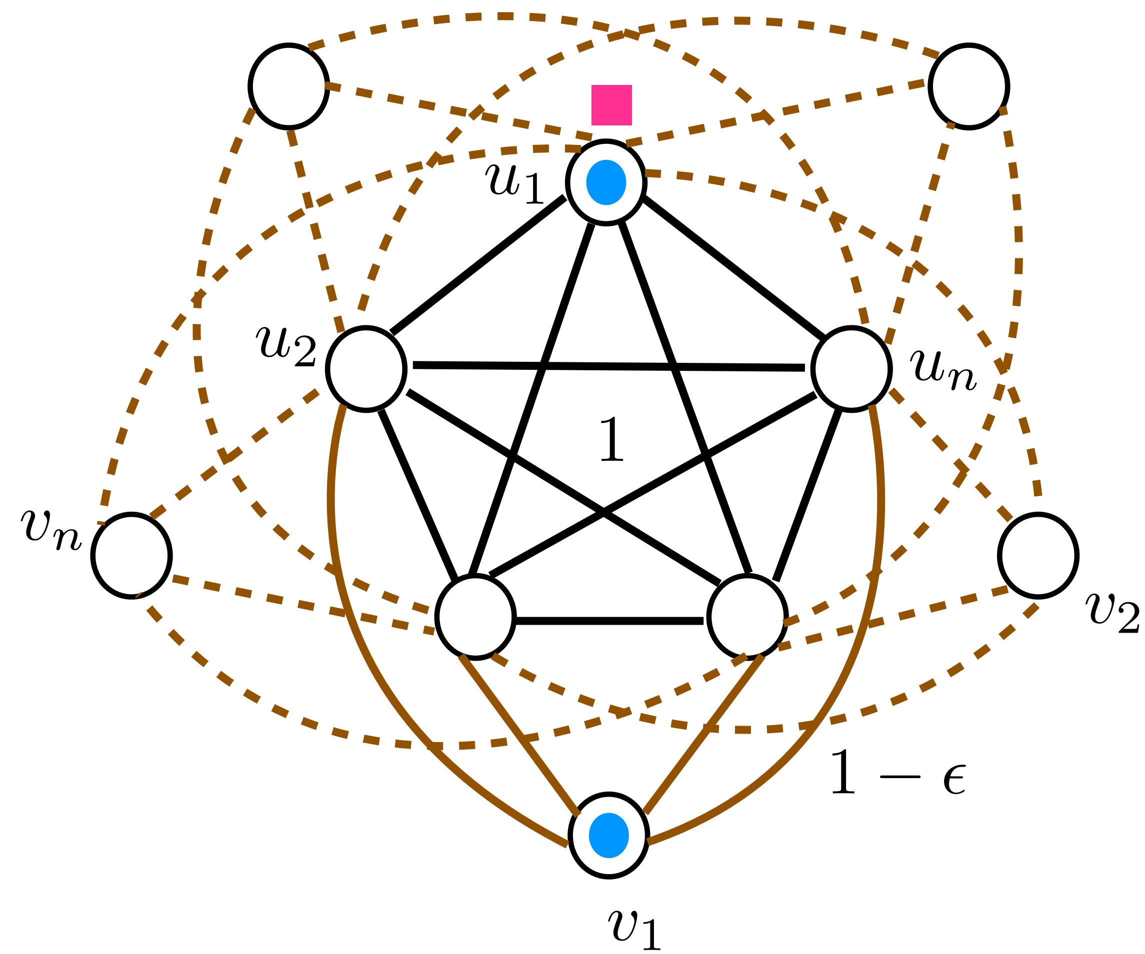


Deterministic GSP mechanisms

- Case 1: facility is opened in U

  All agents but the first one move to v_1. The facility is not opened at v_1

$$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$

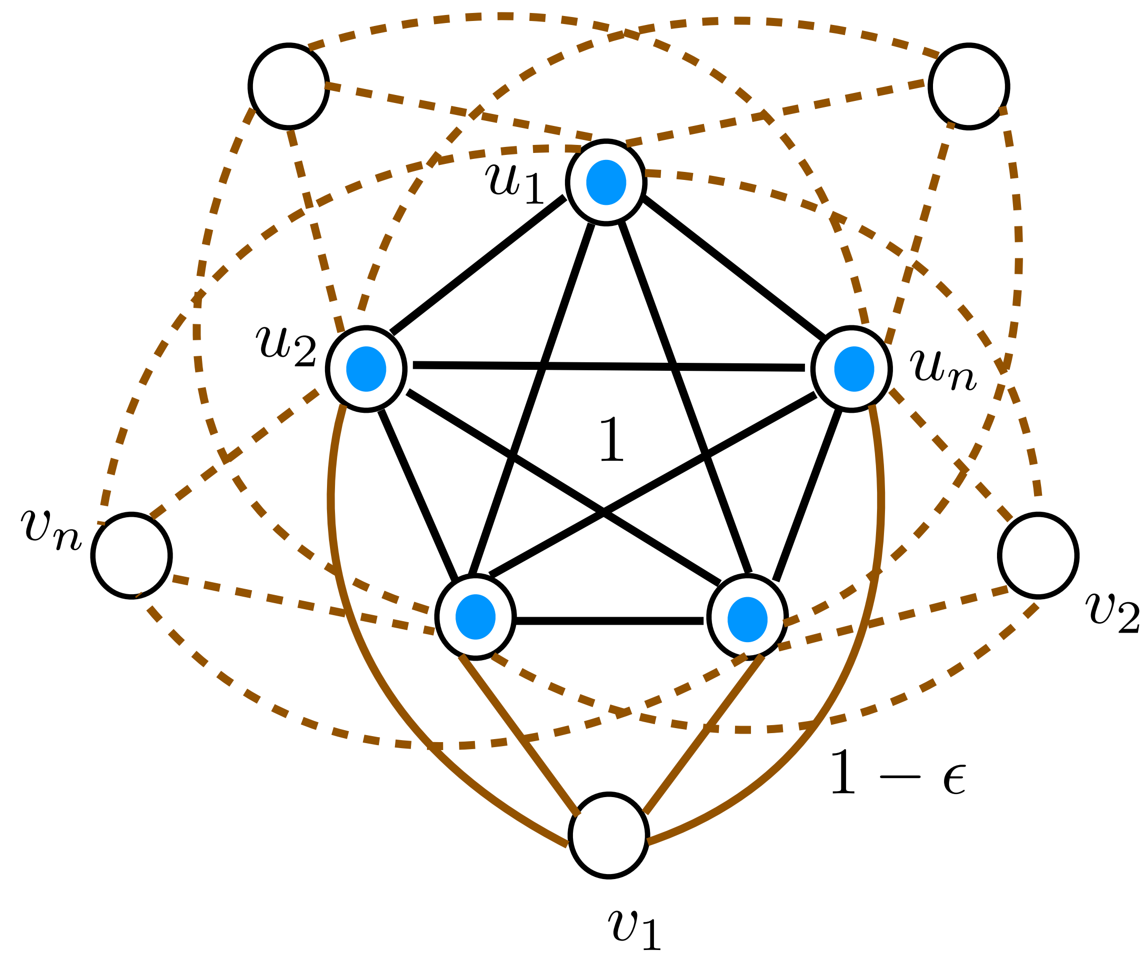


Deterministic GSP mechanisms

- Case 1: facility is opened in U

  All agents but the first one move to v_1. The facility is not opened at v_1

$$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$

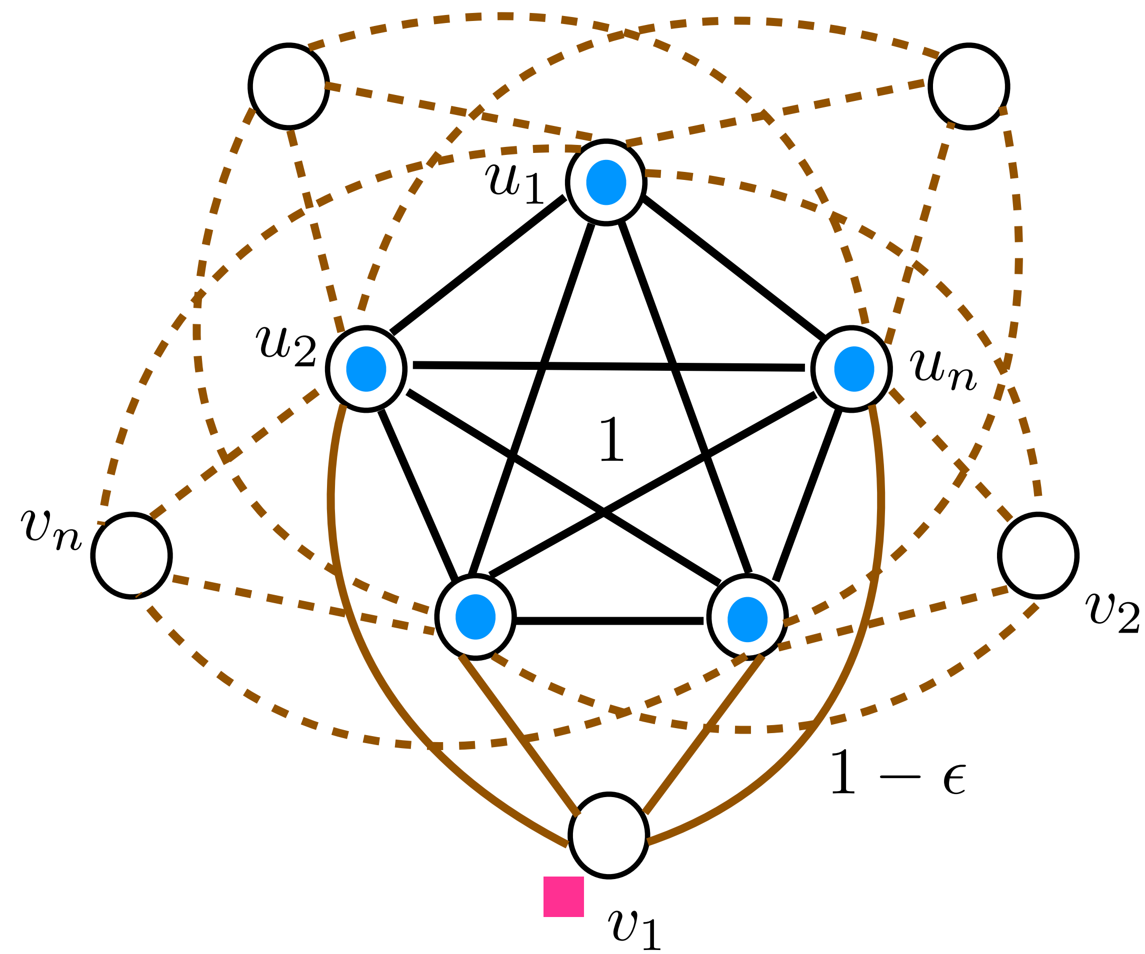


Deterministic GSP mechanisms

- Case 1: facility is opened in U

  All agents but the first one move to v_1. The facility is not opened at v_1

  $$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$



- Case 2: facility is opened in V

  All agents but the last one move to v_n.

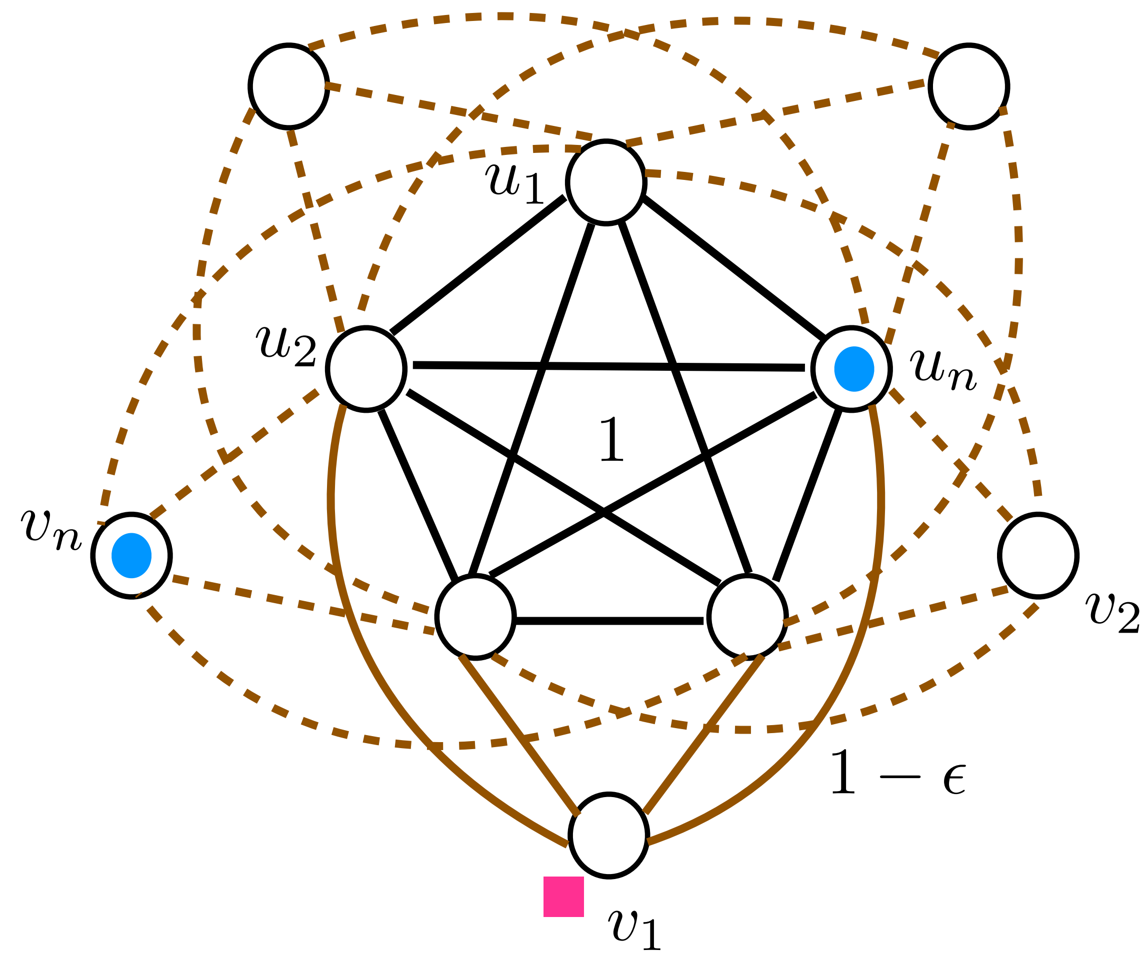
  Again, $$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$

Deterministic GSP mechanisms

- Case 1: facility is opened in U

  All agents but the first one move to v_1. The facility is not opened at v_1

$$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$



- Case 2: facility is opened in V

  All agents but the last one move to v_n .

Again, $$\frac{(n-1)(1-\epsilon)}{2-\epsilon} \approx \frac{n-1}{2}$$

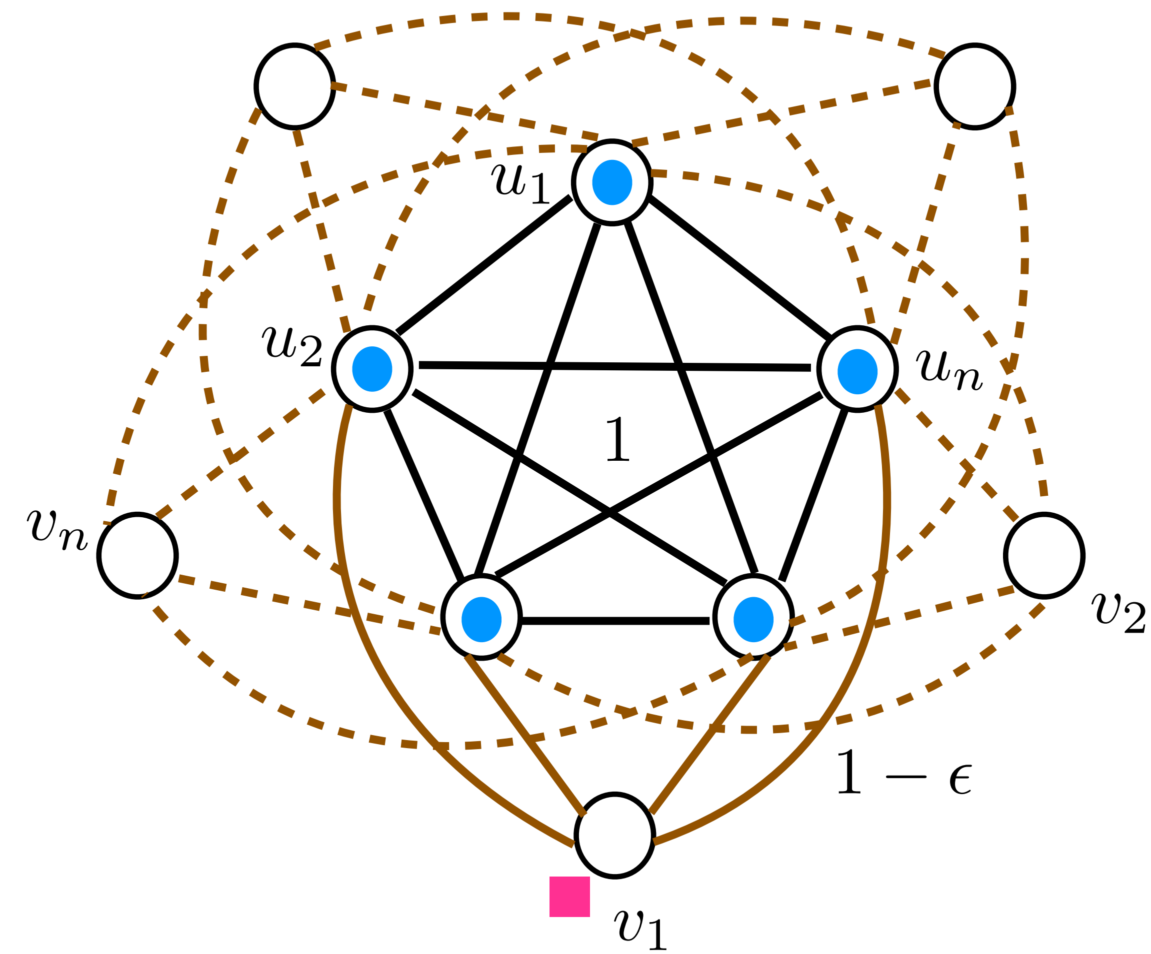
Weakness

- The argument does not carry.
  - An mechanism opens facility at v_n, u_1 with prob $1-\delta, \delta$

    prevent agents $2, \ldots, n-1$ from collaborating.

    old cost of agent 2: $1-\epsilon$

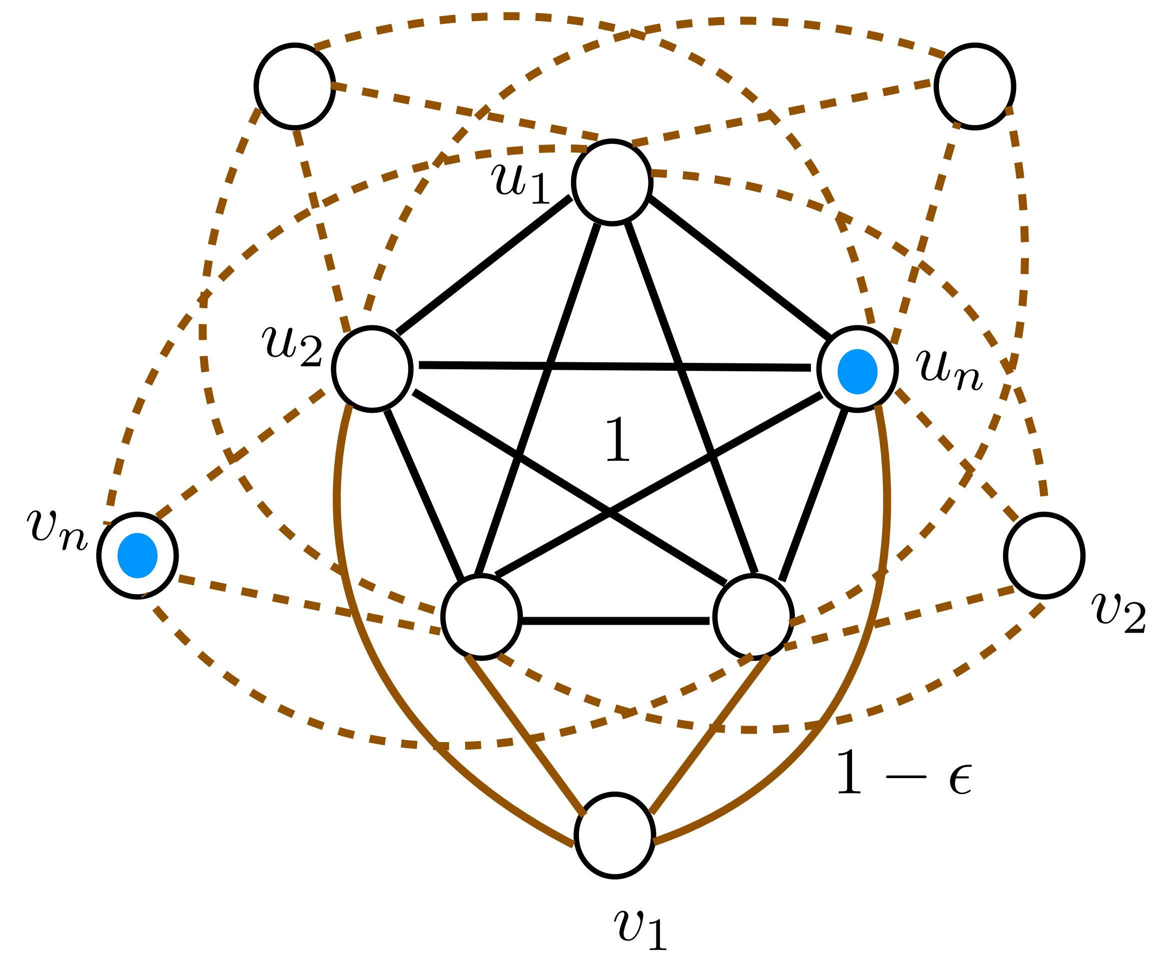
    new cost of agent 2: $(1-\delta)\cdot(1-\epsilon)+\delta\cdot 1$



Weakness

- The argument does not carry.
  - An mechanism opens facility at v_n, u_1 with prob $1-\delta, \delta$

    prevent agents $2, \ldots, n-1$ from collaborating.



old cost of agent 2: $1-\epsilon$

new cost of agent 2: $(1-\delta)\cdot(1-\epsilon)+\delta\cdot 1$

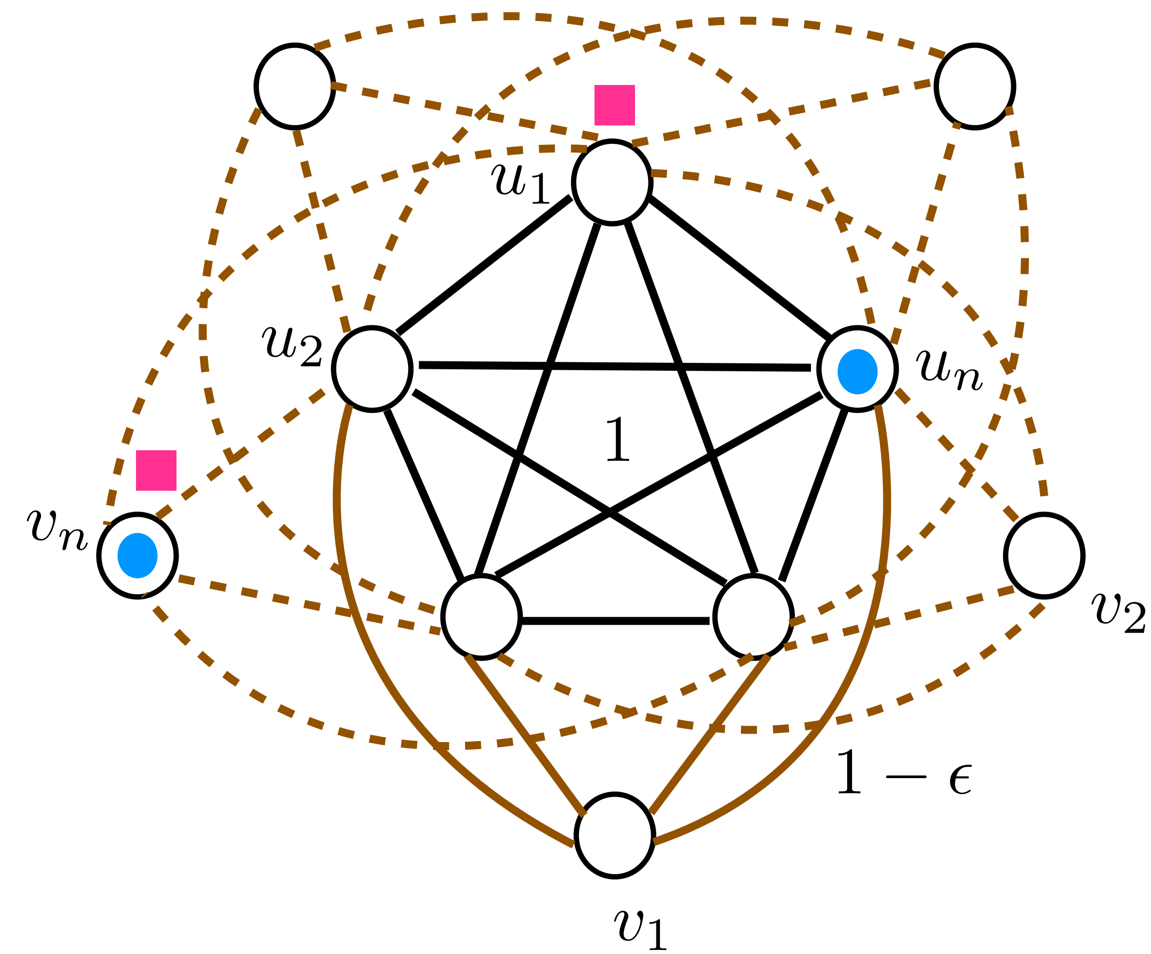
Weakness

- The argument does not carry.
  - An mechanism opens facility at v_n, u_1 with prob $1-\delta, \delta$

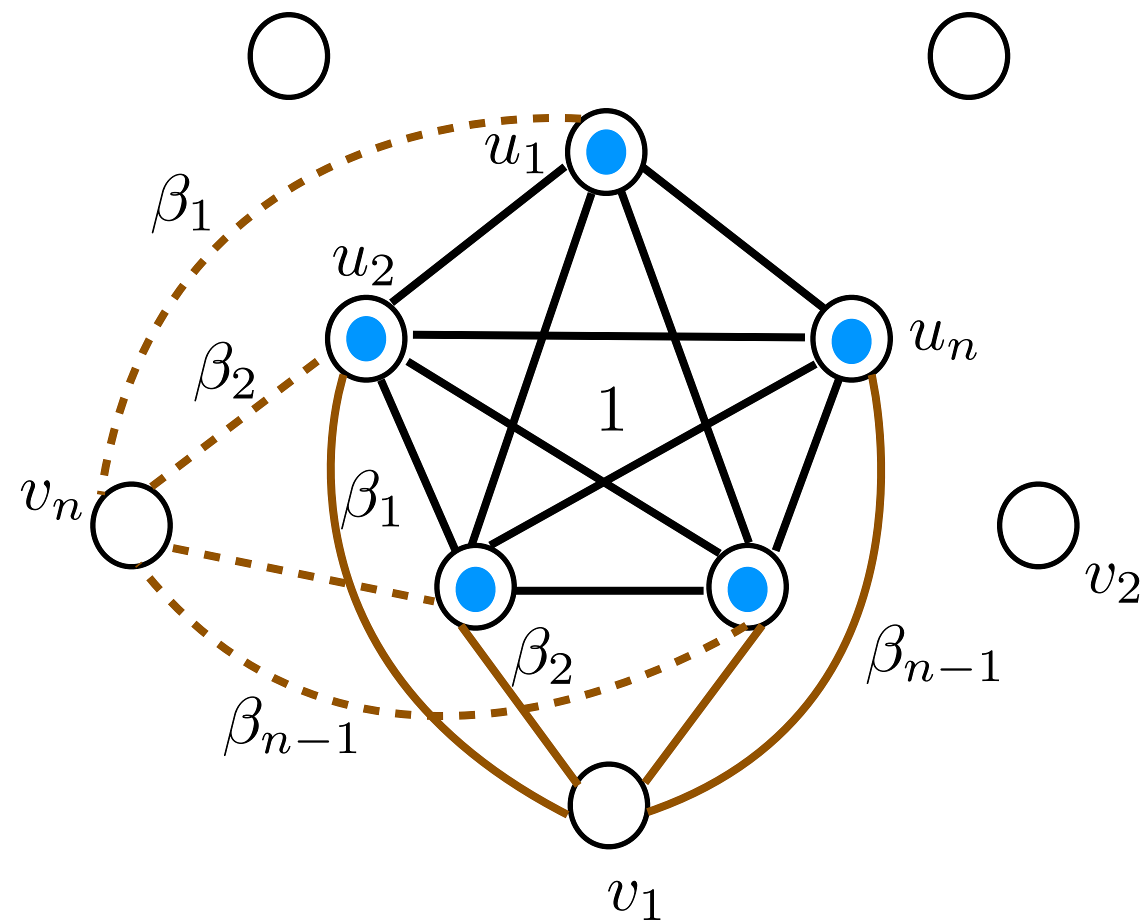
    prevent agents $2, \ldots, n-1$ from collaborating.

    old cost of agent 2: $1-\epsilon$

    new cost of agent 2: $(1-\delta)\cdot(1-\epsilon)+\delta\cdot 1$

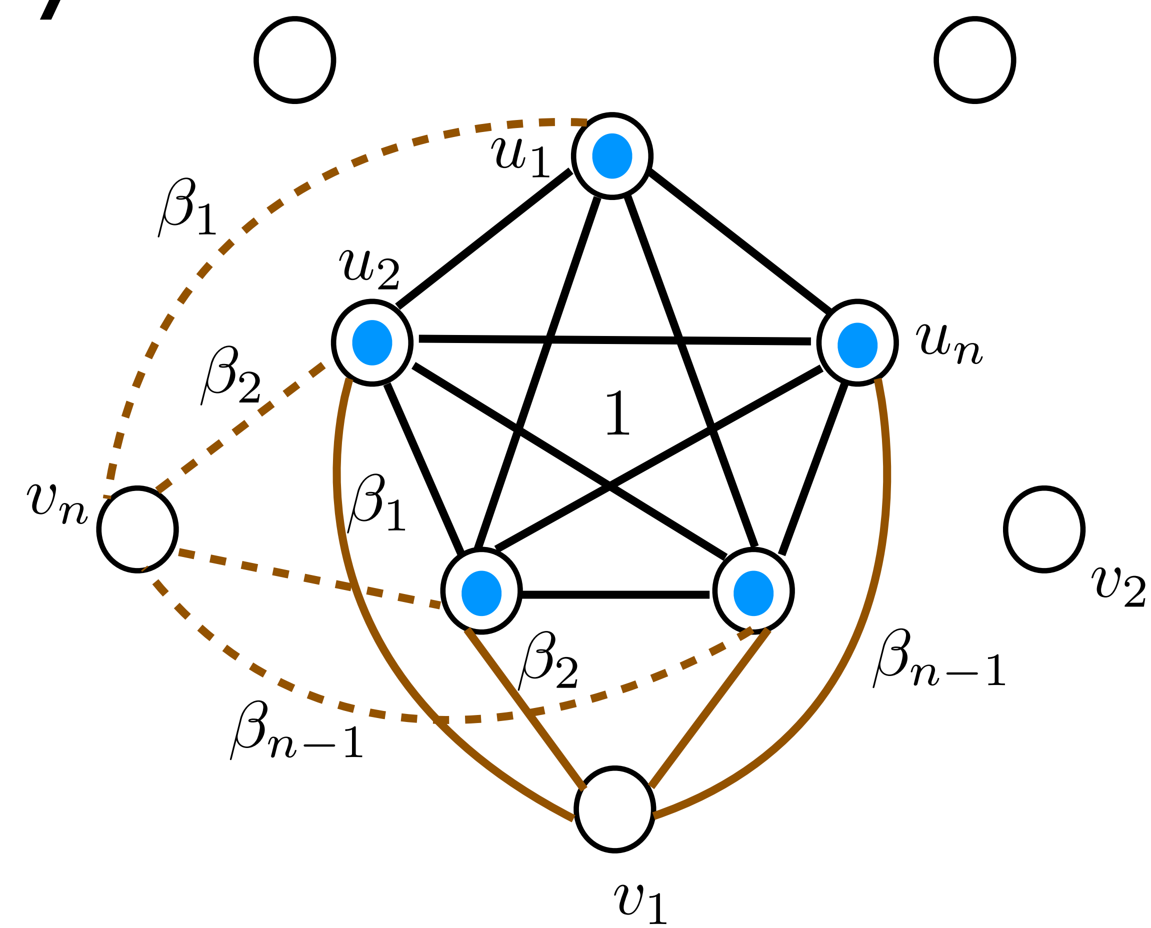


Solution



Solution

- Need: symmetry + asymmetry.

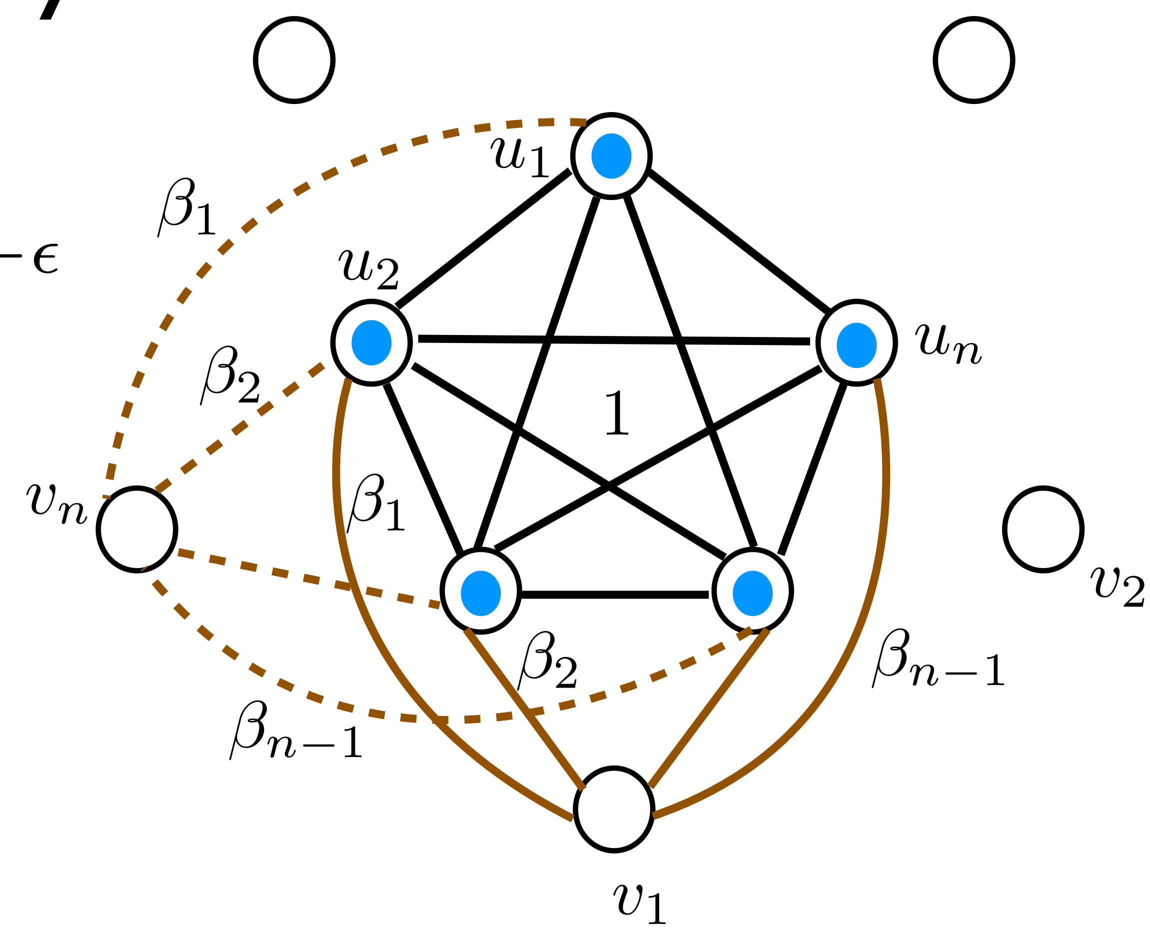


Solution

- Need: symmetry + asymmetry.
  - modify brown edges.

$$2n^{-\epsilon} > \beta_1 > \beta_2 > \ldots > \beta_{n-1} > n^{-\epsilon}$$

  - define the length in circular permutation

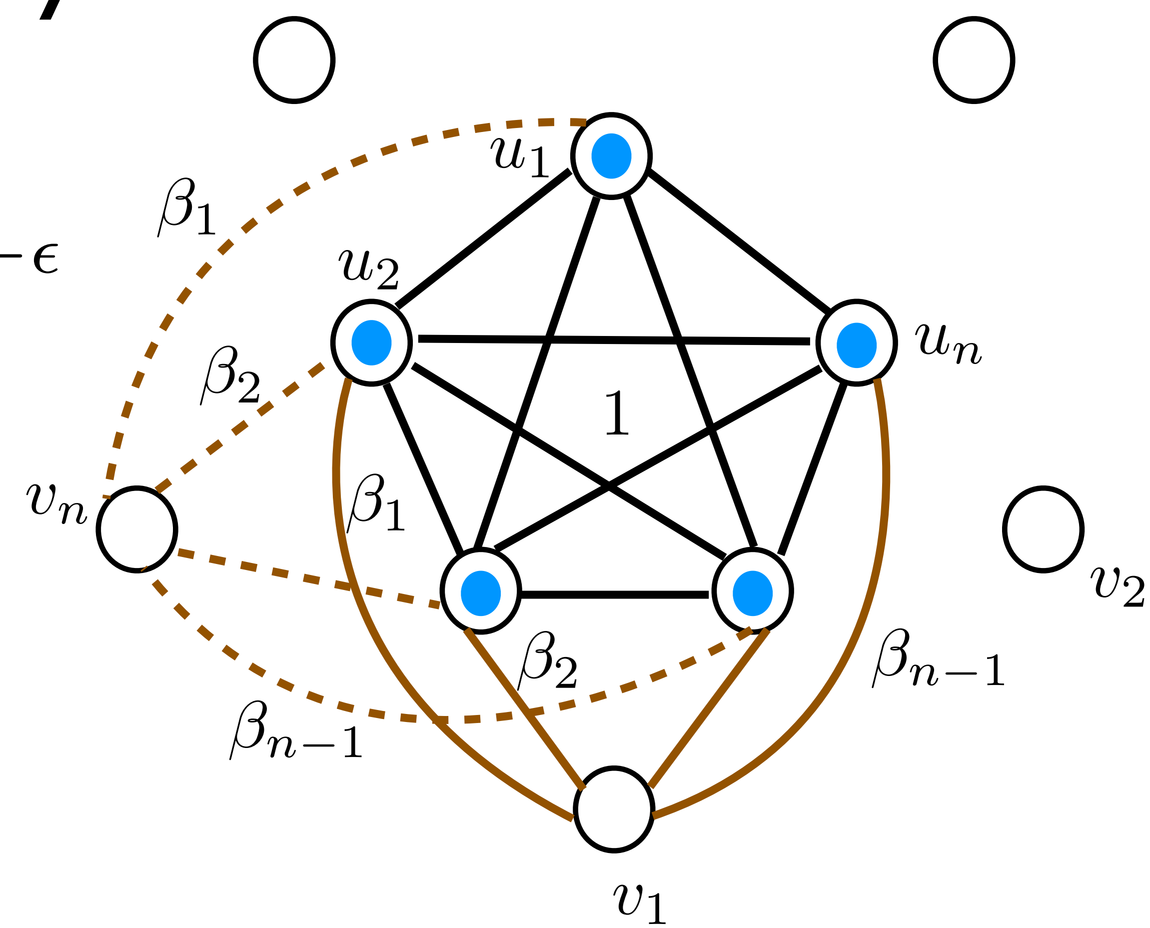


Solution

- Need: symmetry + asymmetry.
  - modify brown edges.

$$2n^{-\epsilon} > \beta_1 > \beta_2 > \ldots > \beta_{n-1} > n^{-\epsilon}$$

  - define the length in circular permutation

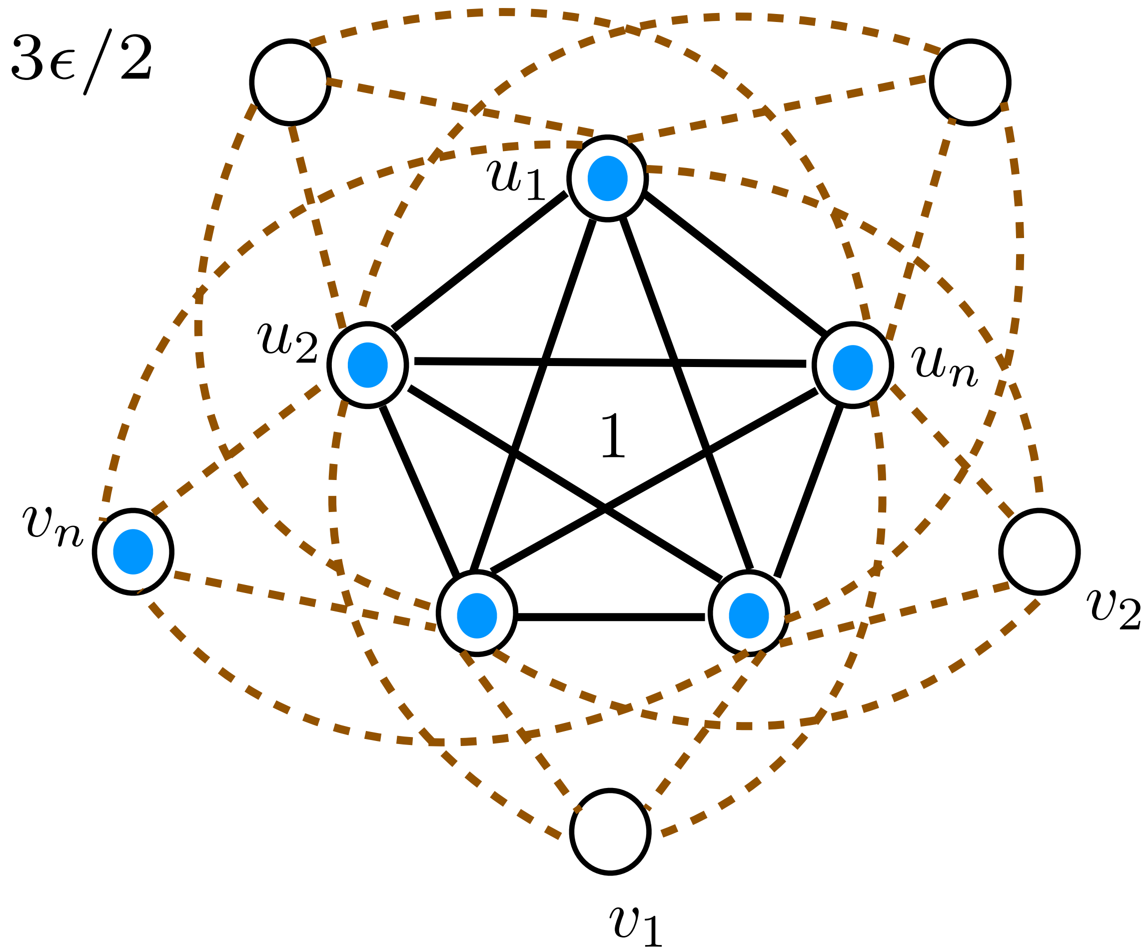


- Why it works?
  - harder to prevent agents from collaborating.
  - amplify the gap by recursive construction

Lower bound construction

- Lemma (informal):

$$\mathbb{P}[f((x)) = v_n] < 1 - \epsilon \left(\frac{\log n}{\log \log n} \right)^{3\epsilon/2}$$

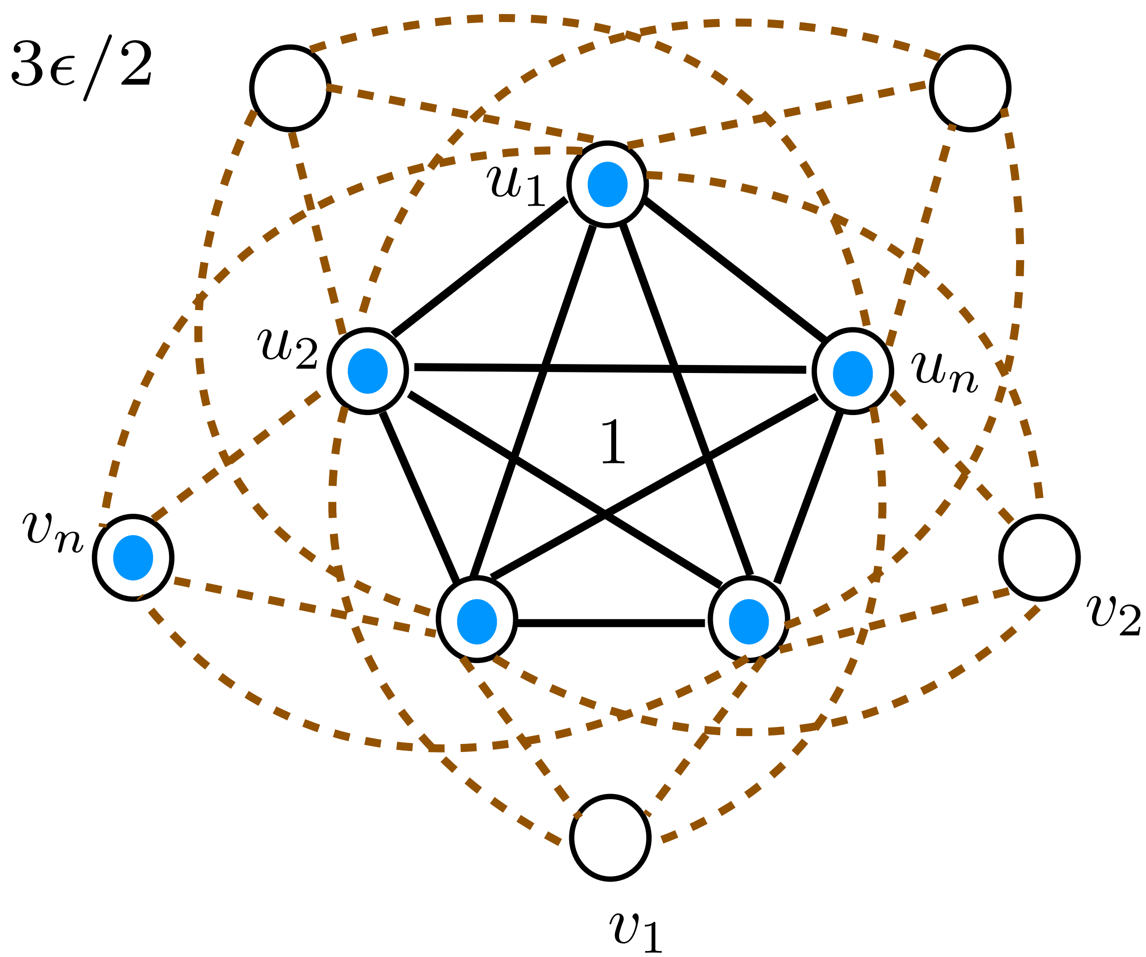


Lower bound construction

- Lemma (informal):

$$\mathbb{P}[f((x)) = v_n] < 1 - \epsilon \left(\frac{\log n}{\log \log n} \right)^{3\epsilon/2}$$

- Recursive construction of multiple levels.

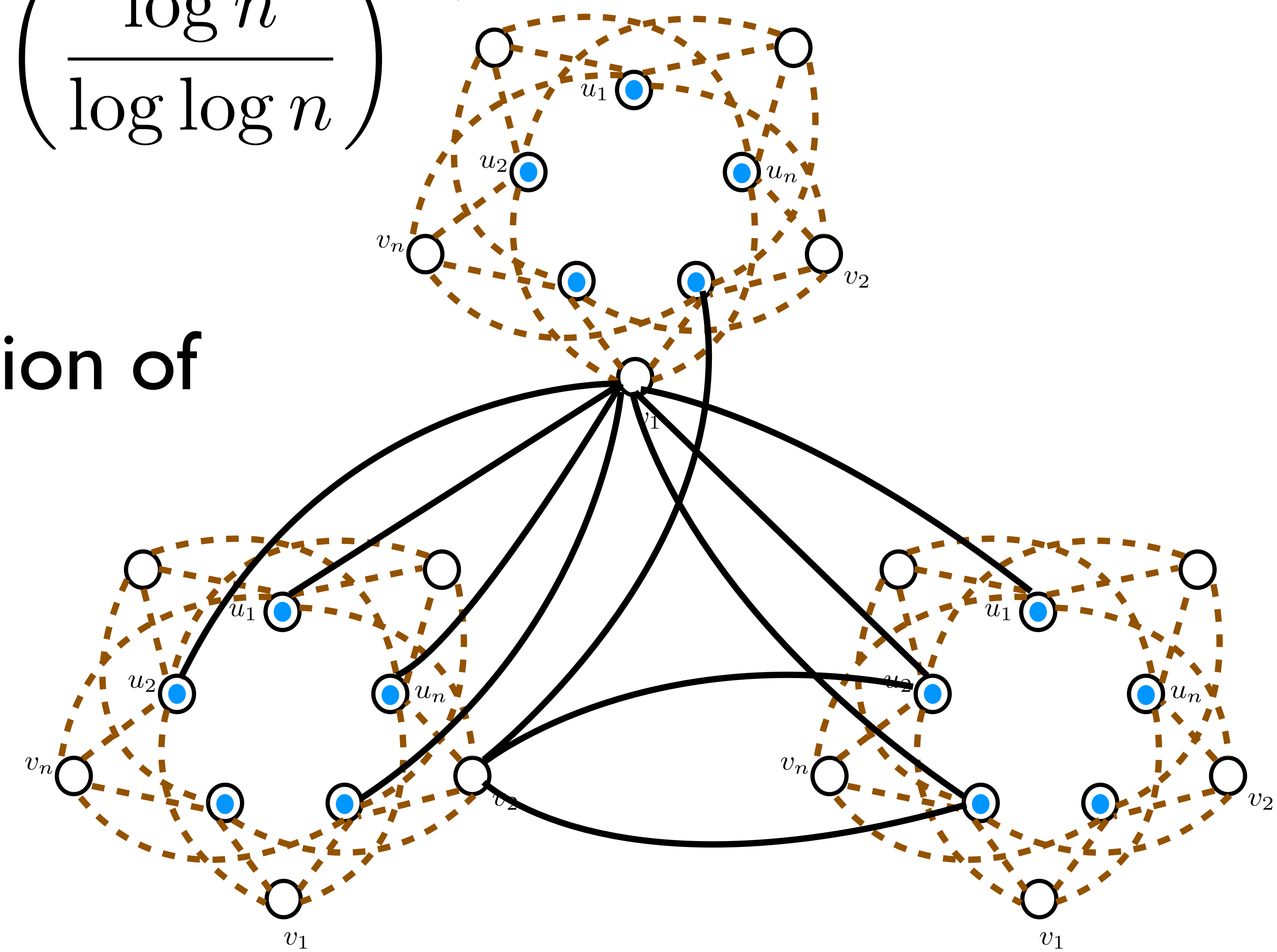


Lower bound construction

- Lemma (informal):

$$\mathbb{P}[f((x)) = v_n] < 1 - \epsilon \left(\frac{\log n}{\log \log n} \right)^{3\epsilon/2}$$

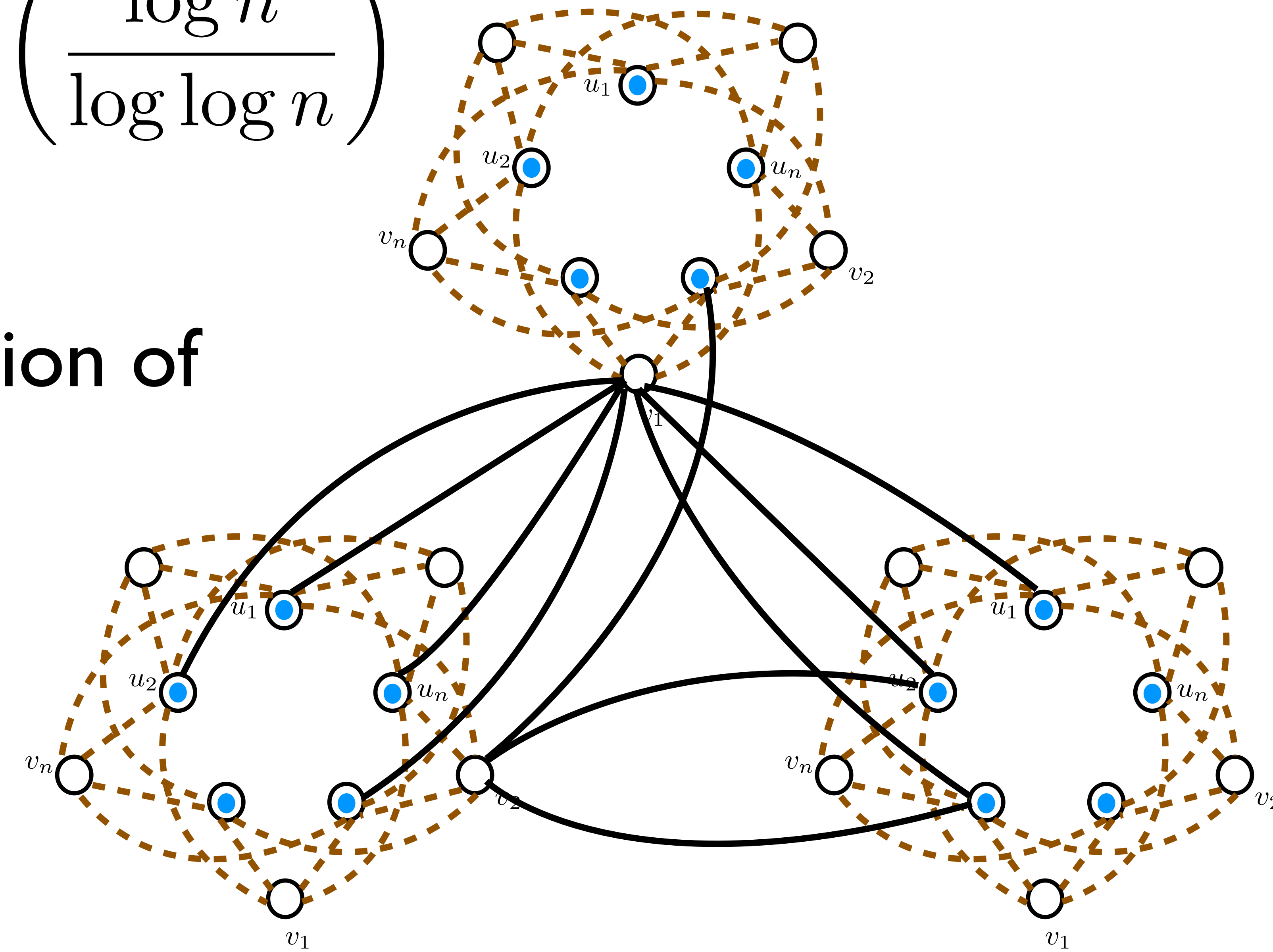
- Recursive construction of multiple levels.



Lower bound construction

- Lemma (informal):

$$\mathbb{P}[f((x)) = v_n] < 1 - \epsilon \left(\frac{\log n}{\log \log n} \right)^{3\epsilon/2}$$



- Recursive construction of multiple levels.

- Lemma (informal):

$$\mathbb{P}[f((x)) = v_n] < 1 - n^{3\epsilon/2}$$

Randomized SP mechanisms

- Mechanism (Random dictatorship): open the facility at x_i with probability $1/n$.
  - ☑ strategy-proof but not GSP
  - ☑ 2-approximation

Randomized SP mechanisms

- Mechanism (Random dictatorship): open the facility at x_i with probability $1/n$.
  - ☑ strategy-proof but not GSP
  - ☑ 2-approximation

Can we do better?

Randomized SP mechanisms

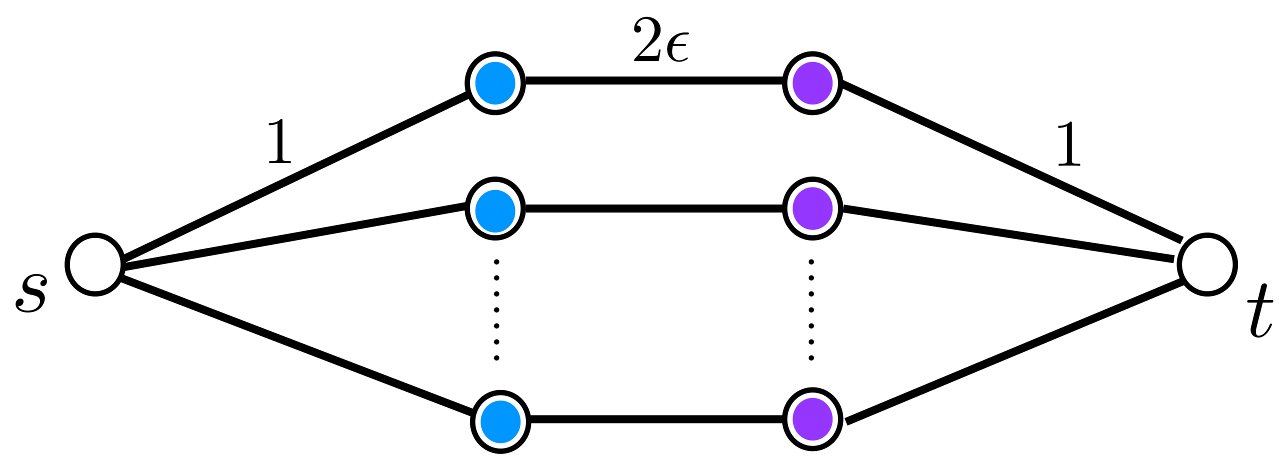
- Mechanism (Random dictatorship): open the facility at x_i with probability $1/n$.
  - strategy-proof but not GSP
  - 2-approximation

Can we do better?

- Thm:
  - no randomized SP mechanism is better than $\left(2 - \frac{4}{n^{1/3}}\right)$ approximation.

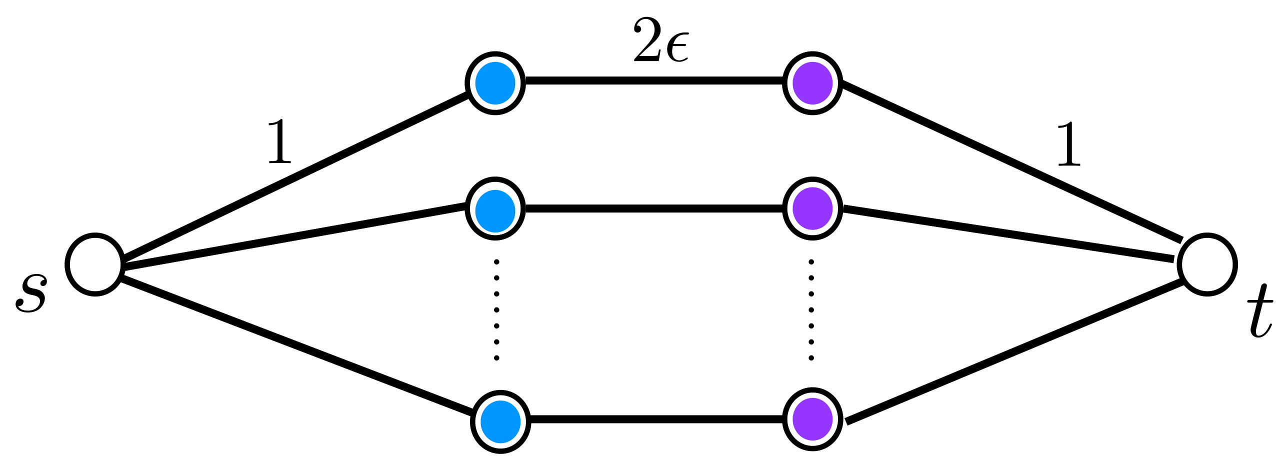
Randomized SP mechanism

- Idea



Randomized SP mechanism

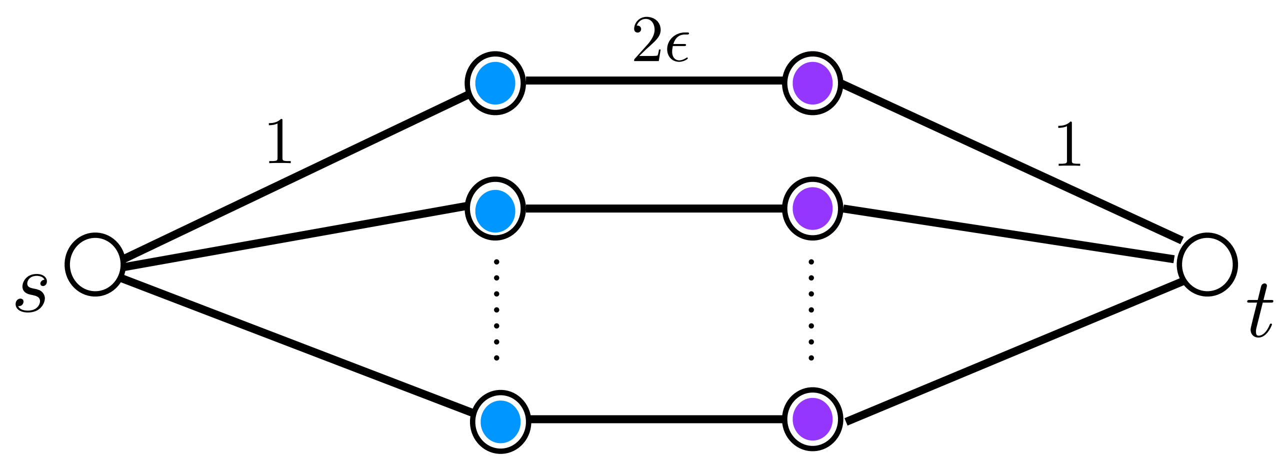
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t

Randomized SP mechanism

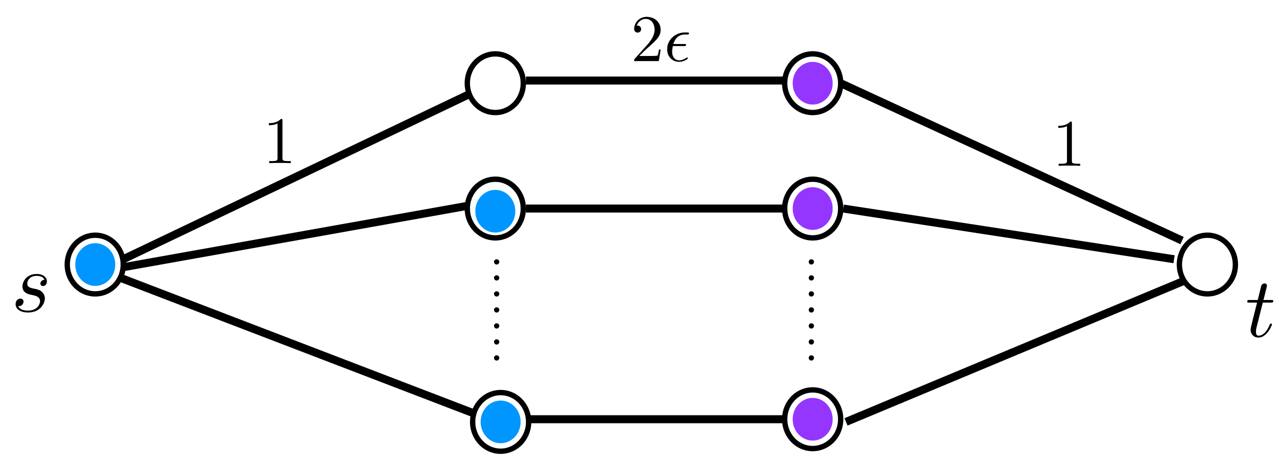
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1 + \epsilon$

Randomized SP mechanism

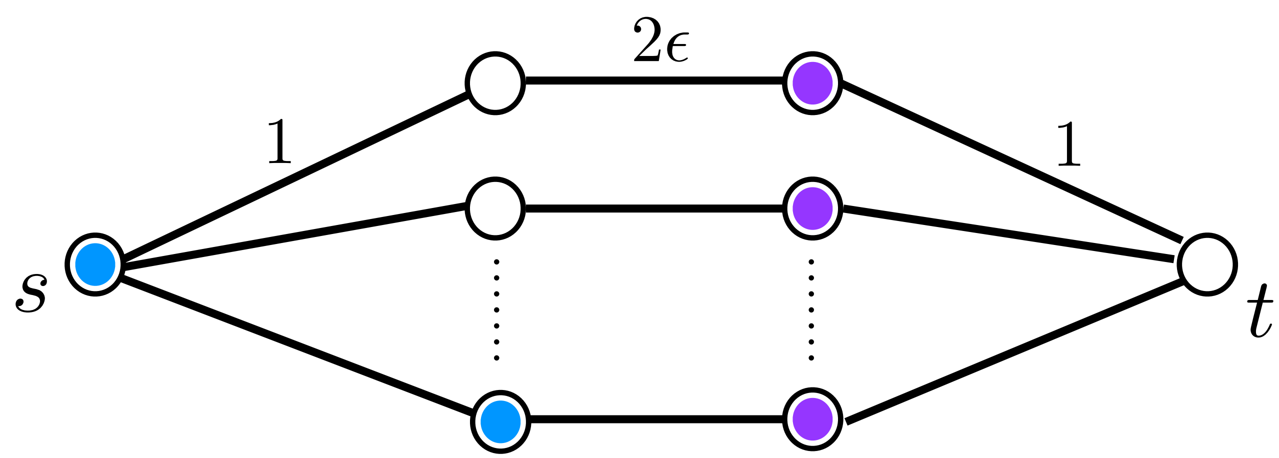
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1+\epsilon$

Randomized SP mechanism

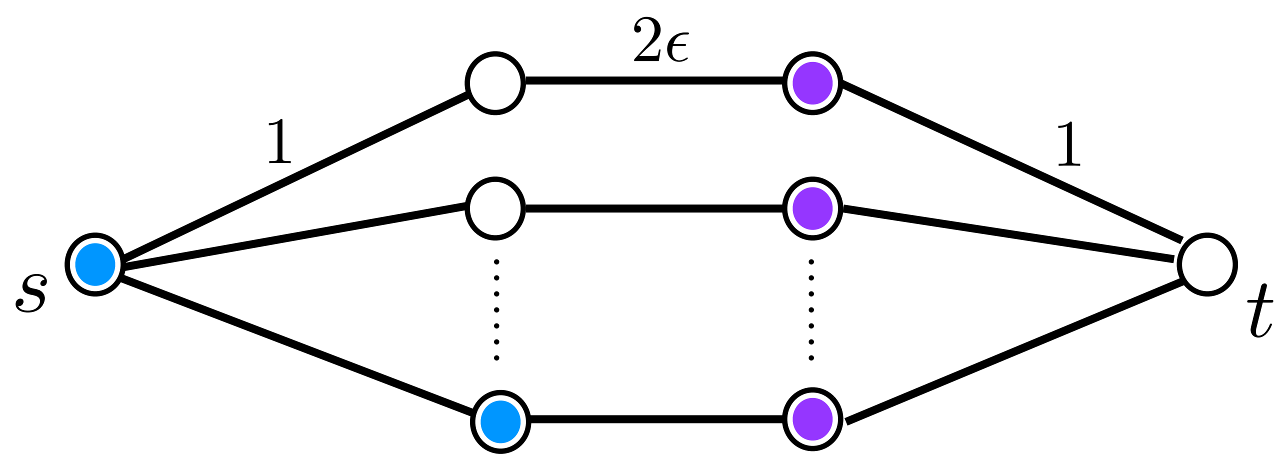
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1 + \epsilon$

Randomized SP mechanism

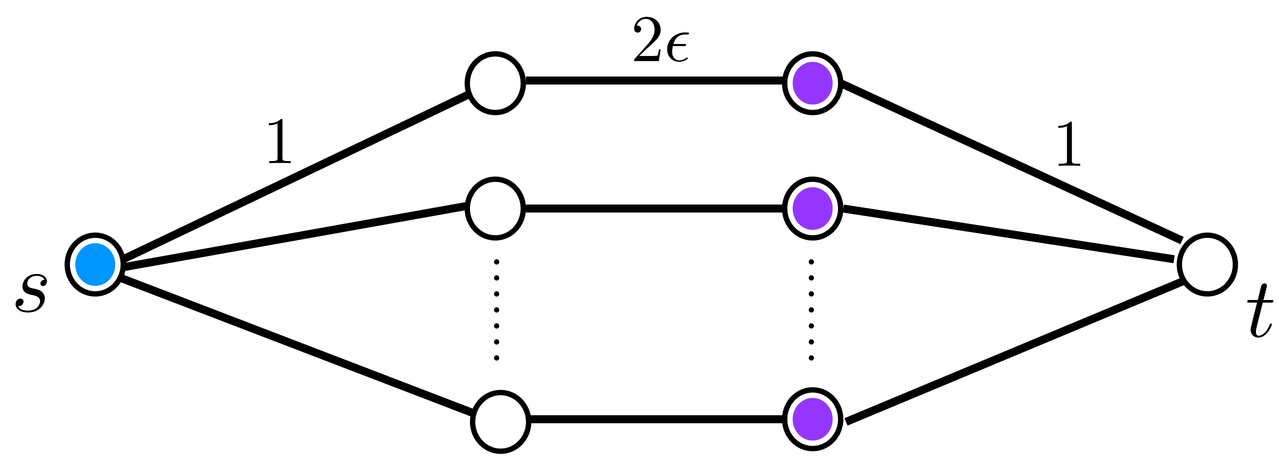
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1 + \epsilon$
  - Move appropriate blue agents to s, by strategy-proof argue that $p_s \approx p_t \approx 1/2$.

Randomized SP mechanism

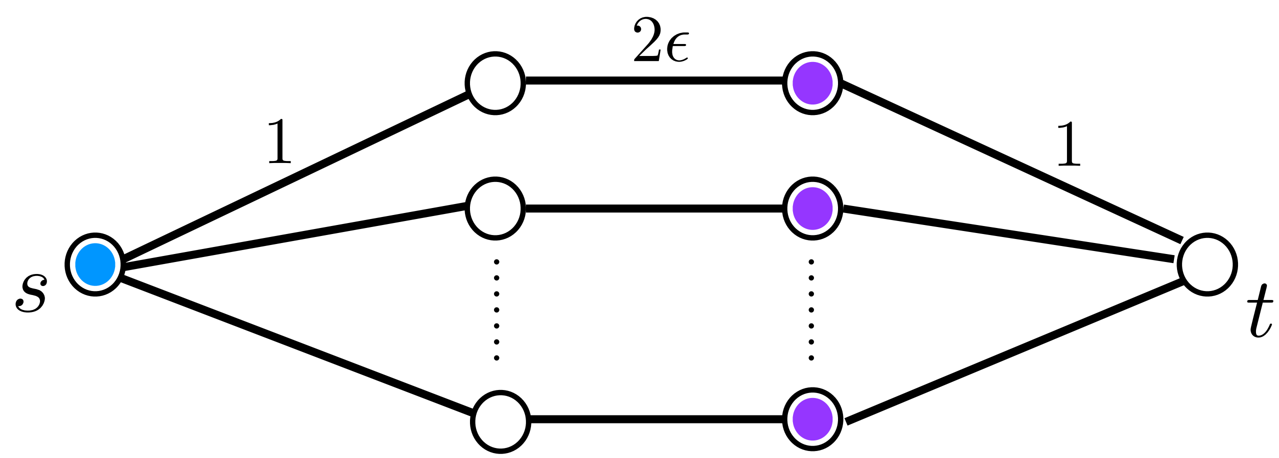
- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1+\epsilon$
  - Move appropriate blue agents to s, by strategy-proof argue that $p_s \approx p_t \approx 1/2$.

Randomized SP mechanism

- Idea

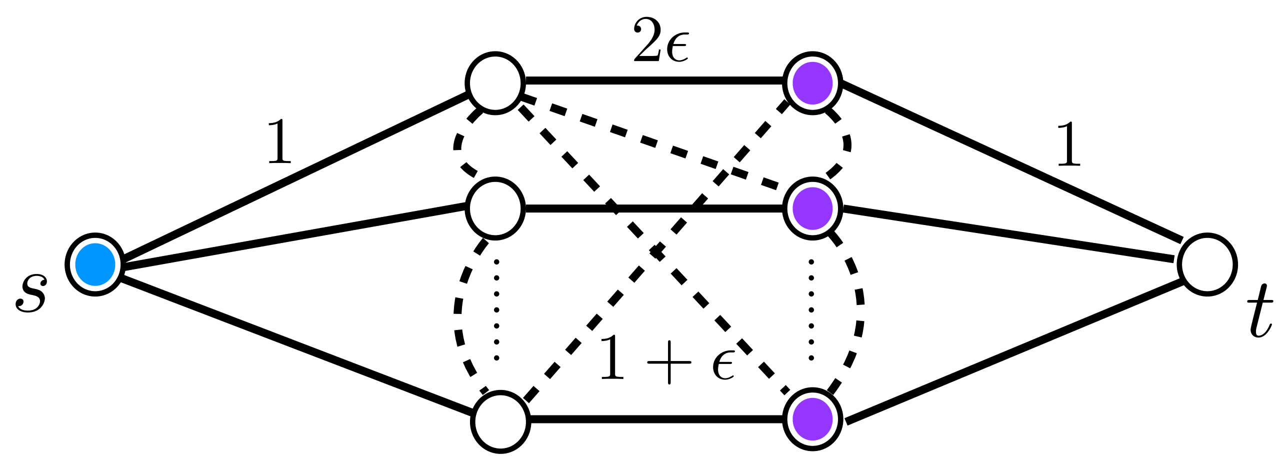


  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1 + \epsilon$
  - Move appropriate blue agents to s, by strategy-proof argue that $p_s \approx p_t \approx 1/2$.

Then: $$\frac{1/2 \cdot (1 + 2 + 2\epsilon) + 1/2 \cdot (1 + 2\epsilon)}{1 + 2\epsilon} \approx 2$$

Randomized SP mechanism

- Idea



  - Initially, "good" mechanism has prob. $1/2$ at s, t
  - There exists an agent, let say blue, with cost at least $1+\epsilon$
  - Move appropriate blue agents to s, by strategy-proof argue that $p_s \approx p_t \approx 1/2$.

Then: $$\frac{1/2 \cdot (1+2+2\epsilon) + 1/2 \cdot (1+2\epsilon)}{1+2\epsilon} \approx 2$$

Conclusion

- ☑ Complete characterization of performance of randomized (G)SP mechanisms.

- ☐ Open a constant facilities:
  - easy in term of optimization.
  - SP mechanism with bounded ratio?

Thank you!