Fundamental Computer Science
Analysis of Vertex Cover

Denis Trystram
MoSIG1 and M1Info – University Grenoble-Alpes

March, 2021
Agenda

- A detailed example: Vertex Cover
Presentation of the Vertex Cover problem

Covering the vertices of a graph by some of its edges.

- **VC**
  - **Input.** A graph $G = (V, E)$ given for instance by its adjacency matrix and an integer $Q$
  - **Question.** Is there a set $V'$ with at most $Q$ vertices that are covering $G$?

This means that each edge of $G$ has at least one of its extremities in $V'$. 
Example
Final covering with 4 sommets
We first verify that \( VC \in \mathcal{NP} \).

Verifier

- generate non-deterministically a set of vertices
- verify that this set is a covering

The verifier is non-deterministic polynomial.
We transform a (positive) instance (positive) of 3SAT into a (positive) instance of VC.

$n$ variables $p_i$

$C_1 \land C_2 \land \cdots \land C_m$

where $C_i = x_{i,1} \lor x_{i,2} \lor x_{i,3}$

where $x_{i,j}$ is a literal on the \{p_1, p_2, \cdots, p_n\}
Construction of the associated graph (the vertices)

- A pair of vertices between each propositional variable $p_i$ and $\neg p_i$
- A triplet of vertices for each clause $C_i$

The number of vertices is thus equal to $2n + 3m$
Construction of the associated graph (the edges)

- An edge between each pair $p_i$ and $\neg p_i$
- An edge between each of the three vertices of the triangles $C_i$
- An edge between each $x_{i,j}$ and the vertex $p$ or $\neg p$ depending of the literal

The number of edges is thus equal to $3m + 3m + n$
Construction of the associated graph (the constant $Q$)

$$Q = 2m + n$$
Construction of the associated graph (the constant $Q$)

\[ Q = 2m + n \]

Of course, this transformation is polynomial.
Example (with 4 variables)

\[(p_2 \lor \neg p_1 \lor p_4) \land (\neg p_3 \lor \neg p_2 \lor \neg p_4)\]
Example (2 triangles)

\[(p_2 \lor \neg p_1 \lor p_4) \land (\neg p_3 \lor \neg p_2 \lor \neg p_4)\]
Example (associations between $x$ and $p$)

$$(p_2 \lor \neg p_1 \lor p_4) \land (\neg p_3 \lor \neg p_2 \lor \neg p_4)$$
The final graph

\[(p_2 \lor \neg p_1 \lor p_4) \land (\neg p_3 \lor \neg p_2 \lor \neg p_4)\]
First, let remark that the previous transformation is polynomial.

We prove now that

3SAT is satisfiable iff the transformed graph has a covering at most $2m + n$. 
Proof ($\Rightarrow$)

- Let consider a positive instance of 3SAT
- There exists an interpretation function $\gamma$ that affects $TRUE$ to all the clauses $C_i$.
The covering is derived from this function, it contains the following vertices.
  - The vertices $p_i$ for which $\gamma$ affects $TRUE$ and the $\neg p_i$ for which it affects $FALSE$.
  - 2 vertices for each triangle, chosen such as $\gamma$ affects $TRUE$ to the non-chosen vertex\(^1\).

The size of this covering is exactly $n + 2m$.

\(^1\)that is always possible since as all the clauses are $TRUE$, the function affects at least one literal to $TRUE$ in each clause.
Proof ($\Rightarrow$)

- Let consider a positive instance of 3SAT
- There exists an interpretation function $\gamma$ that affects $TRUE$ to all the clauses $C_i$. The covering is derived from this function, it contains the following vertices.
  - The vertices $p_i$ for which $\gamma$ affects $TRUE$ and the $\neg p_i$ for which it affects $FALSE$.
  - 2 vertices for each triangle, chosen such as $\gamma$ affects $TRUE$ to the non-chosen vertex\(^1\).

The size of this covering is exactly $n + 2m$.

- We should also verify that all the edges are well covered. This is easy by considering successively each type of edge.

---

\(^1\)that is always possible since as all the clauses are $TRUE$, the function affects at least one literal to $TRUE$ in each clause
A solution of VC on the previous example

$$(p_2 \lor \neg p_1 \lor p_4) \land (\neg p_3 \lor \neg p_2 \lor \neg p_4)$$

$$(p_2 = 1, \neg p_4 = 1)$$ is solution (whatever the value of $p_1$ and $p_3$).
Consider now that the graph has a covering of size $2m + n$

it defines an interpretation function that makes the formula $TRUE$. 
Consider now that the graph has a covering of size $2m + n$
and it defines an interpretation function that makes the formula $TRUE$.

- A covering of size $2m + n$ contains necessarily a vertex of each pair $(p_i, \neg p_i)$ and two vertices in each triangle.
Consider now that the graph has a covering of size $2m + n$ it defines an interpretation function that makes the formula $TRUE$.

- A covering of size $2m + n$ contains necessarily a vertex of each pair $(p_i, \neg p_i)$ and two vertices in each triangle. It could not have more because in this case, the size will be more than $2m + n$. 
Consider now that the graph has a covering of size $2m + n$ it defines an interpretation function that makes the formula $TRUE$.

- A covering of size $2m + n$ contains necessarily a vertex of each pair $(p_i, \neg p_i)$ and two vertices in each triangle. It could not have more because in this case, the size will be more than $2m + n$.

- The interpretation function is the one that affects $TRUE$ to $p_i$ if it is in the covering set and $FALSE$ if $\neg p_i$ is not. The existence of this covering implies that the corresponding 3SAT formula is $TRUE$ since one of the $(p_i, \neg p_i)$ is covered for each triangle (clause).