Fair division of jewels

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HomeWork **Maths for Computer Science** – *MOSIG 1 – 2022*

Guidelines

This work can be done by groups of maximum 3 students, but each student should send a personal report in pdf at

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Please, indicate the names of the other members of this group and detail clearly the credit of each (percentage in designing, proving, writing code, do experiments, etc.).

I don't encourage you to look at Internet, but in case, indicate also all sources you used.

Use your professional email address and clearly indicate:

subject: **Homework MCS**

The file should be named **Homework-name.pdf**

The strict deadline is **novembre 14, 23:59**, a penalty will be applied in case of delay.

Presentation

The goal of this homework is to study mathematical properties of a combinatorial problem. Here it is:

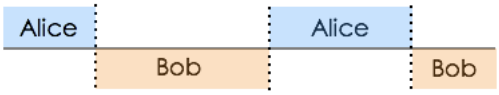
Two thieves Alice and Bob meet after a robbery. They want to share the loot (that is a necklace made of jewels) in a fair way.

Formal definition of the problem

Let us consider an open necklace composed of n jewels of k types (n_1 rubies, n_2 diamonds, n_3 sapphires, etc.). Here is an example for $k = 3$.

The problem is to determine a *fair* division while **cutting the necklace into several pieces of consecutive jewels with the minimum number of cuts**.

In other words, each thief leaves with one or more pieces with the same amount of jewels as shown below:



Preliminary: Two types of jewels

- Prove that the problem of $k = 2$ types of jewels can be solved in at most 2 cuts.

Let us now turn to the case of $k = 3$ types of jewels.

Warm-up: a negative preliminary results for $k = 3$

- Show through a counter-example that two cuts are not sufficient to divide a necklace composed of 3 types of jewels.

Some positive results for $k = 3$

Let us start by a simple strategy.

Assume n_1, n_2 and n_3 are even. Both Alice and Bob should receive each half of the n_i.

- A possible strategy for a repartition between the two thieves is to start (say by the left of the necklace) and cut the necklace at the first piece of consecutive jewels of *maximum possible number of jewels* and give it to Alice. More precisely, it is cut just before we are reaching $\frac{n_i}{2} + 1$ for some $1 \leq i \leq 3$.

  Then, continue the repartition by switching to the second thief with another piece of maximum possible size for Bob.

  Turn again the same way with the maximum remaining piece to Alice, and so on...

- Run this strategy on the previous example.
- Formalize it in a pseudo-code.
- Build an example to show that this strategy could lead to more than 3 cuts.
- Provide a mathematical analysis to obtain a lower bound of the number of cuts of the previous strategy[1].
- Design your own strategy and analyze its *performance* in term of cuts[2].

[1] that could depend on n

[2] you are encourage to do experiments to assess the analysis

Special case of necklace with k pairs of jewels

If the number of jewels of each type is equal to 2, the problem of cutting the necklace with a minimum number of cuts can be solved for any number of types k.

- Describe and analyze an *optimal* solution.