Fundamental Computer Science
Sequence 1: Turing Machines
Classical extensions

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MoSIG1-M1info, 2021
Objective of the session
Study the most common extensions of Turing machines.

See Pierre Wolper, Introduction à la calculabilité or any related book.
Extensions of the Turing Machine

We have already presented an extension:

- write in the tape and move left or right at the same time
- modify the definition of the transition function

\[
\begin{align*}
\text{initial:} & \quad \text{from} \ (K \setminus H) \times \Gamma \ \text{to} \ K \times (\Gamma \cup \{\leftarrow, \rightarrow\}) \\
\text{extended:} & \quad \text{from} \ (K \setminus H) \times \Gamma \ \text{to} \ K \times \Gamma \times \{\leftarrow, \rightarrow\}
\end{align*}
\]
We have already presented an extension:

- **write** in the tape and **move** left or right at the same time
- modify the definition of the transition function
  
  **initial**: from \((K \setminus H) \times \Gamma\) to \(K \times (\Gamma \cup \{\leftarrow, \rightarrow\})\)
  
  **extended**: from \((K \setminus H) \times \Gamma\) to \(K \times \Gamma \times \{\leftarrow, \rightarrow\}\)

- if the **extended** Turing Machine halts on input \(w\) after \(t\) steps, then the **initial** Turing Machine halts on input \(w\) after at most \(2t\) steps
A $k$-tape Turing Machine ($M$) is a sextuple $(K, \Sigma, \Gamma, \delta, s, H)$, where $K$, $\Sigma$, $\Gamma$, $s$ and $H$ are as in the definition of the ordinary Turing Machine, and $\delta$ is a transition function

$$
\text{from } (K \setminus H) \times \Gamma^k \text{ to } K \times (\Gamma \cup \{\leftarrow, \rightarrow\})^k
$$

\[\begin{array}{c}
\begin{array}{c}
q_4 \\
q_3 \\
q_2 \\
\bullet \rightarrow q_1 \\
qu_0
\end{array}
\begin{array}{c}
\cdots \square \square \ 1 \ 1 \ 0 \ \square \ \square \ \cdots \\
\cdots \square \ \square \ a \ a \ b \ a \ b \ \square \ \square \ \cdots \\
\cdots \square \ \square \ a \ b \ a \ b \ b \ a \ \square \ \square \ \cdots
\end{array}
\end{array} \]
A $k$-tape Turing Machine $(M)$ is a sextuple $(K, \Sigma, \Gamma, \delta, s, H)$, where $K$, $\Sigma$, $\Gamma$, $s$ and $H$ are as in the definition of the ordinary Turing Machine, and $\delta$ is a transition function

$$\text{from} \quad (K \setminus H) \times \Gamma^k \quad \text{to} \quad K \times (\Gamma \cup \{\leftarrow, \rightarrow\})^k$$

$$(\text{from} \quad (K \setminus H) \times \Gamma^k \quad \text{to} \quad K \times \Gamma^k \times \{\leftarrow, \rightarrow\}^k)$$
Multiple tapes

**Theorem**

Every $k$-tape, $k > 1$, Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$ has an equivalent single tape Turing Machine $M' = (K', \Sigma', \Gamma', \delta', s', H')$.

If $M$ halts on input $w \in \Sigma^*$ after $t$ steps, then $M'$ halts on input $w$ after $O(t(|w| + t))$ steps.

Sketch of the proof:

- $M'$ simulates $M$ in a single tape
- # is used as delimiter to separate the contents of different tapes
- dotted symbols are used to indicate the actual position of the head of each tape
  - for each symbol $\sigma \in \Gamma$, add both $\sigma$ and $\cdot$ in $\Gamma'$
- use the same set of halting states
Multiple tapes

Sketch of the proof:

\[
\begin{array}{c}
M \\
\end{array}
\]

\[
\begin{array}{c}
M' \\
\end{array}
\]
Multiple tapes

Sketch of the proof:

\[ M' = \text{“On input } w = w_1 w_2 \ldots w_n:\]

1. Format the tape to represent the \( k \) tapes:
   \[
   \#ullet w_1 w_2 \ldots w_n \# \square \# \square \# \ldots \#
   \]

2. For each step that \( M \) performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of \( M \).
Multiple tapes

Sketch of the proof:

$M' = \text{“On input } w = w_1w_2 \ldots w_n:\n1. \text{ Format the tape to represent the } k \text{ tapes:}
   \text{#}w_1w_2 \ldots w_n\text{# □ # □ # … □ #}
2. \text{ For each step that } M \text{ performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of } M.
3. \text{ If at any point there is a need to move a virtual head outside the area marked for the corresponding tape, then shift right the contents of all tapes succeeding.”}
Sketch of the proof:

\[ M' = "\text{On input } w = w_1w_2 \ldots w_n:\]

1. Format the tape to represent the \( k \) tapes:
   \[
   \#\bullet w_1w_2 \ldots w_n\# \square \# \square \# \ldots \#
   \]

2. For each step that \( M \) performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of \( M \).

3. If at any point there is a need to move a virtual head outside the area marked for the corresponding tape, then shift right the contents of all tapes succeeding.”

What is the number of steps for \( M' \)?
Multiple tapes

Sketch of the proof:

$M' = \text{“On input } w = w_1w_2 \ldots w_n:}$

1. Format the tape to represent the $k$ tapes:
   \[ \#w_1w_2 \ldots w_n\# \square \# \square \# \ldots \# \]

2. For each step that $M$ performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of $M$.

3. If at any point there is a need to move a virtual head outside the area marked for the corresponding tape, then shift right the contents of all tapes succeeding.”

What is the number of steps for $M'$?

1. $O(|w|)$

2. & 3. $O(|w| + t) \text{ per step} \Rightarrow O(t(|w| + t))$ in total
   - size of the tape no more than $O(|w| + t)$
Multiple tapes: conclusion

The multiple tape Turing Machine is not more powerful!!
Multiple tapes: conclusion

The multiple tape Turing Machine is **not** more powerful!!

... but it is more easy to construct and to understand!
The multiple tape Turing Machine is not more powerful!!

... but it is more easy to construct and to understand!

... and it can be used to simulate functions in an easier way
(a function can use one or more not used tapes)
Multiple tapes: example with $k = 2$ tapes

\[
\begin{align*}
\text{Operation} & \quad \text{Symbol} & \quad \text{Symbol} \\
> R^{1,2} & \quad \sigma^1 \neq \square & \quad \sigma^2 \\
\text{extend notation} & \quad R^{1,2} & \quad \sigma^1
\end{align*}
\]

Initial state: $\square \quad w \quad \square$

After (1) $\square \quad w \quad \square$ transformation:

$\square \quad w \quad \square$
Multiple tapes: example with $k = 2$ tapes

- **extend notation:**
  - $R^{1,2}$: move the head of both tapes to the right
  - $\sigma^2$ (as a state): write the symbol $\sigma$ in tape 2
  - $\sigma^2$ (as a label): if the head of tape 2 reads the symbol $\sigma$
Multiple tapes: example with $k = 2$ tapes

$\sigma^2 \neq \square \rightarrow \sigma^1$

$\sigma^1 \neq \square \rightarrow \sigma^2$

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\[
\begin{array}{c|c|c}
\text{Initially} & \text{tape 1} & \text{tape 2} \\
\hline
w \square & \square & \square \\
\end{array}
\]

after (1)
Multiple tapes: example with $k = 2$ tapes

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<th>initially</th>
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<tbody>
<tr>
<td>after (1)</td>
<td>$\sqcup w \sqcup$</td>
<td>$\sqcup$</td>
</tr>
<tr>
<td>after (2)</td>
<td>$\sqcup w \sqcup$</td>
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<tr>
<td>initially</td>
<td>$\sqsubseteq w$</td>
<td>$\sqsubseteq$</td>
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<tr>
<td>after (1)</td>
<td>$\sqsubseteq w\sqsubseteq$</td>
<td>$\sqsubseteq w\sqsubseteq$</td>
</tr>
<tr>
<td>(2)</td>
<td>$\sqsubseteq w\sqsubseteq$</td>
<td>$\sqsubseteq w\sqsubseteq$</td>
</tr>
<tr>
<td>at the end</td>
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Multiple tapes: example with $k = 2$ tapes

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<td>(2)</td>
<td>$\sqcup$\textit{w}$\sqcup$</td>
<td>$\textit{w}$\textit{w}$\sqcup$</td>
</tr>
<tr>
<td>at the end</td>
<td>$\textit{w}$\textit{w}$\sqcup$ $\textit{w}$\textit{w}$\sqcup$</td>
<td>$\textit{w}$\textit{w}$\sqcup$</td>
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Multiple tapes: example with $k = 2$ tapes

$\mu > R^{1,2} \sigma^1 \neq \Box \rightarrow \sigma^2$

$\mu^1$

$L^2 \mu R^{1,2} \sigma^2 \neq \Box \rightarrow \sigma^1$

(1)

(2)

▶ extend notation:

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<td>$\Box w \Box w \Box$</td>
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transforms $w$ to $w \Box w$
Another extension: Multiple heads

Definition (informal)

- at each step all heads can read/write/move
- we need a convention if two heads try writing at the same place
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Theorem

Every multiple head Turing Machine $M$ has an equivalent single head Turing Machine $M'$.

The simulation by $M'$ of $M$ on an input $w$ which leads to a halting state takes time quadratic to the size of the input $|w|$ and the number of steps $t$ that $M$ performs.

Proof (sketch):
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Proof (sketch):

- scan the tape twice
  1. find the symbols at the head positions (which transition to follow?)
  2. write/move the heads according to the transition
- same arguments as before for the number of steps
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Proof (another one):

\[
\begin{array}{cccccccccc}
\vdots & \square & m & y & \square & i & n & p & u & t & \square & \cdots \\
\hline
\wedge & & & & & & & & & & & \\
\hline
\wedge & & & & & & & & & & & \\
\hline
\wedge & & & & & & & & & & & \\
\hline
\wedge & & & & & & & & & & & \\
\end{array}
\]
Give a Machine Turing with two heads that transforms the input $\square w$ to $\square w \sqcup w$.

- extend notation:
  - $\sigma, \overline{\sigma}, \overline{\overline{\sigma}}$: the position of the 1st, 2nd and both heads, respectively
  - $R^{1,2}$: move both heads on the right
  - $\sigma^2$ (as a state): write in the position of head 2 the symbol $\sigma$
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Multiple heads: example

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Unbounded tapes

What happens if the tape is bounded in one direction?
Unbounded tapes

What happens if the tape is bounded in one direction?

**Theorem**

*Every two-direction unbounded tape Turing Machine $M$ has an equivalent single-direction unbounded tape Turing Machine.*
Two-dimensional tape

Definition (informal)

- move the head left/right/up/down
Two-dimensional tape

Definition (informal)

▸ move the head left/right/up/down

Why?
Two-dimensional tape

Definition (informal)
- move the head left/right/up/down

Why?
- for example, to represent more easily two-dimensional matrices

Theorem
Every two-dimensional tape Turing Machine \( M \) has an equivalent single-dimensional tape Turing Machine \( M' \).

The simulation by \( M' \) of \( M \) on an input \( w \) which leads to a halting state takes time polynomial to the size of the input \( |w| \) and the number of steps \( t \) that \( M \) performs.

Proof (sketch):
- use a multiple tape Turing Machine
- each tape corresponds to one line of the two-dimensional memory
Two-dimensional tape

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- move the head left/right/up/down

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Discussion

- We can even combine the presented extensions and still **not** get a stronger model.
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- We can even combine the presented extensions and still not get a stronger model.

- Observation: a computation in the prototype Turing Machine needs a number of steps which is bounded by a polynomial of the size of the input and of the number steps in any of the extended model.