Maths for Computer Science
Computing the sum of cubes

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Training class MoSIG1

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Recall

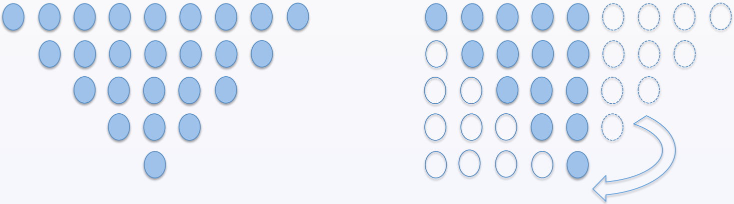
The problem

Definition:
Sum of the first n odd numbers

$$S_n = \sum_{k=0}^{n-1} (2k+1) = \mathbf{?}$$

Method 1: Fubini's principle

The bullets depict the consecutive odd numbers. The arrangement of the bullets gives two ways for counting.



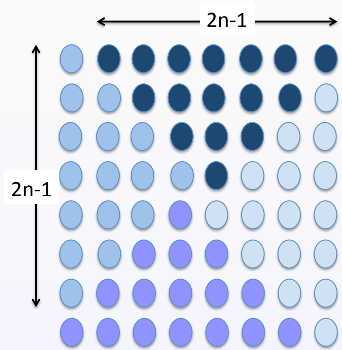
Result: $S_n = n^2$

Method 2: shifted Gauss trick

$$
\begin{aligned}
\Delta_n + \Delta_{n-1} &= 1 + \boxed{\begin{matrix}2\\ +1\end{matrix}} + \boxed{\begin{matrix}3\\ +2\end{matrix}} + \ldots + \boxed{\begin{matrix}n\\ +n-1\end{matrix}} \\
&= 1 + 3 + 5 + \ldots + (2n-1)
\end{aligned}
$$

Method 3: an other pictorial proof

We can also imagine a construction which uses four copies of S_n that exactly correspond to a $2n$ by $2n$ square as depicted below.



Therefore, $4S_n = (2n)^2$ and $S_n = n^2$.

The problem

Definition:
Sum of the first n cubes

$$C_n = \sum_{k=1}^{n} k^3 = \textbf{?}$$

Hints

- Determine the **asymptotic behavior** of the summation
- Apply the **undetermined coefficient** method
- Compute on the first ranks and prove the expected result by **induction** on n

Determine the asymptotic behavior

A **similar** analysis to the **sum of squares** leads us to:

$$C_n = \Theta\left(\frac{n^4}{4}\right)$$

A first brute force method

Similarly to the problem of computing the **sum of squares**, we may use the method of **undetermined coefficients**:

Let us write $C_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3 + +\alpha_4 n^4$

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Let us write $C_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3 + +\alpha_4 n^4$

This method requires to solve a 5 by 5 system of linear equations (that can be simplified into a 4 by 4 system since $\alpha_0 = 0$).

Let us take the time to **observe the sum** more carefully...

Computing C_n on the first ranks

$$
\begin{array}{lcrcr}
C_1 & = & 1 & = & 1 \\
C_2 & = & 1+8 & = & 9 \\
C_3 & = & 9+27 & = & 36 \\
C_4 & = & 36+64 & = & 100 \\
C_5 & = & 100+125 & = & 225 \\
\vdots & & \vdots & & \vdots
\end{array}
$$

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\end{array}
\qquad
\begin{array}{lcr}
\Delta_1 & = & \mathbf{1} \\
\Delta_2 & = & \mathbf{3} \\
\Delta_3 & = & \mathbf{6} \\
\Delta_4 & = & \mathbf{10} \\
\Delta_5 & = & \mathbf{15} \\
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Observation: $C_n = \Delta_n^2$ (prove with **induction**)

An alternative method

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Proposition:

For all n,

$$\sum_{k=1}^{n} k^3 = \Delta_n^2 = \sum_{k=1}^{\Delta_n} (2k-1) \tag{1}$$

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For all n,

$$\sum_{k=1}^{n} k^3 = \Delta_n^2 = \sum_{k=1}^{\Delta_n} (2k-1) \qquad (1)$$

Note:

The left-hand of equation (1) follows from our previous induction proof, and the right-hand follows from the sum of the first odd integers when we sum up to Δ_n instead of summing up to n.

The core problem: sum of cubes from sum of odds

We take the odd integers in order and arrange them into groups whose successive sizes increase by 1 at each step, as follows:

$$\begin{array}{lrrrr}
\text{group 1 (size 1):} & 1, & & & \\
\text{group 2 (size 2):} & 3, & 5, & & \\
\text{group 3 (size 3):} & 7, & 9, & 11, & \\
\text{group 4 (size 4):} & 13, & 15, & 17, & 19 \\
\vdots & & \vdots & &
\end{array} \tag{2}$$

Observation of Table (2)

We observe first that[1] each row (ith group) adds up to i^3

$$\begin{array}{lrrrrl}
\text{group 1 (size 1):} & 1, & & & & : \text{sum} = \ 1^3 \\
\text{group 2 (size 2):} & 3, & 5, & & & : \text{sum} = \ 2^3 \\
\text{group 3 (size 3):} & 7, & 9, & 11, & & : \text{sum} = \ 3^3 \\
\text{group 4 (size 4):} & 13, & 15, & 17, & 19 & : \text{sum} = \ 4^3
\end{array}$$

[1] at least within the illustrated portion of the table

Proof

$$\begin{array}{lrrrr}
\text{group 1:} & 1, & & & \\
\text{group 2:} & 3, & 5, & & \\
\text{group 3:} & 7, & 9, & 11, & \\
\text{group 4:} & 13, & 15, & 17, & 19
\end{array}$$

By construction, the ith group of odd integers in the table consists of the i consecutive odd numbers beginning with the $(\Delta_{i-1} + 1)$th odd number, namely, $2\Delta_{i-1} + 1$.

Proof

group 1:	1,			
group 2:	3,	5,		
group 3:	7,	9,	11,	
group 4:	13,	15,	17,	19

By construction, the *i*th group of odd integers in the table consists of the i consecutive odd numbers beginning with the $(\Delta_{i-1}+1)$th odd number, namely, $2\Delta_{i-1}+1$.

Since consecutive odd numbers differ by 2, this means that the *i*th group (for $i > 1$) consists of the following i odd integers:

$$2\Delta_{i-1}+1,\ 2\Delta_{i-1}+3,\ 2\Delta_{i-1}+5,\ \ldots,\ 2\Delta_{i-1}+(2i-1)$$

The ith group:

$$2\Delta_{i-1} + 1,\ 2\Delta_{i-1} + 3,\ 2\Delta_{i-1} + 5,\ \ldots,\ 2\Delta_{i-1} + (2i - 1)$$

The *sum* of the integers in the ith group (σ_i) equals:

$$\sigma_i \quad =$$

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The *sum* of the integers in the ith group (σ_i) equals:

$$\begin{aligned}\sigma_i &= 2i\Delta_{i-1} + \big(1+3+\cdots+(2i-1)\big)\\ &= 2i\Delta_{i-1} + (\text{the sum of the first } i \text{ odd numbers})\\ &= 2i\Delta_{i-1} + i^2\end{aligned}$$

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By direct calculation, then,

$$\sigma_i = 2i\cdot\frac{i(i-1)}{2} + i^2 = (i^3-i^2) + i^2 = i^3$$

The proof is now completed by concatenating the rows (summing all the groups) of Table (2) and observing the pattern that emerges:

$$(1) \; + \; (3+5) \; + \; (7+9+11) \; + \cdots \;\; = \;\; 1^3 \; + \; 2^3 \; + \; 3^3 \; + \cdots$$

$\square$

Pictorial proof

We now present the relation between sums of perfect cubes and squares of triangular numbers.

This illustration provides a *non-textual* way to understand this result, and it provides a fertile setting for seeking other facts of this type.

Definition:
For all n,

$$1^3 + 2^3 + \cdots + n^3 = \Delta_n^2$$

Proof

We develop an induction that reflects the structure of Table (2).

Base case.

$$1^3 \;=\; 1 \;=\; \Delta_1^2$$

While this first (and obvious) case is enough for the induction, it does not tell us much about the **structure of the problem**. Therefore, we consider also the next step $n = 2$:

$$1^3 + 2^3 \;=\; 9 \;=\; \Delta_2^2$$

Illustration.

$$1^3 + 2^3 \ = \ 9 \ = \ \Delta_2^2$$

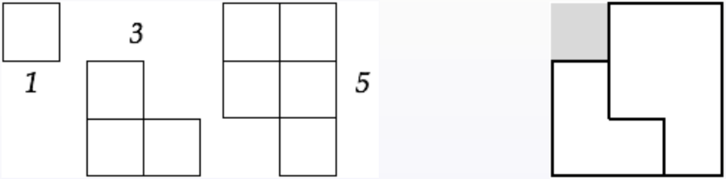

Figure: (Left) the set of group 1 is $\{1\}$ and the set of group 2 is $\{3, 5\}$. (Right) how to form a 3×3 square by pictorially summing the numbers 1, 3, and 5.

Observe that we can fit the shapes from the left side of the figure together to form the $\Delta_2 \times \Delta_2$ square.

Inductive hypothesis. Assume that the target equality holds for all $k < n$; i.e.,

$$1^3 + 2^3 + \cdots + k^3 = \Delta_k^2$$

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If we go one step further, to incorporate group 3, i.e., the set $\{7, 9, 11\}$, into our pictorial summation process, then we discover that mimicking the previous process is a bit more complicated here.

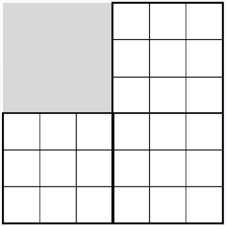
More complicated manipulation required to form the $\Delta_3 \times \Delta_3$ square is a consequence of the odd cardinality of the group-3 set.

We must extend our induction for the cases of odd and even k.

Inductive extension for odd k.

$$\Delta_k^2 \;=\; \Delta_{k-1}^2 \;+\; k^3$$

We write k^3 as $k \times k^2$, and we distribute $k \times k$ square blocks around the $\Delta_{k-1} \times \Delta_{k-1}$ square, as shown below for the case $k = 3$.

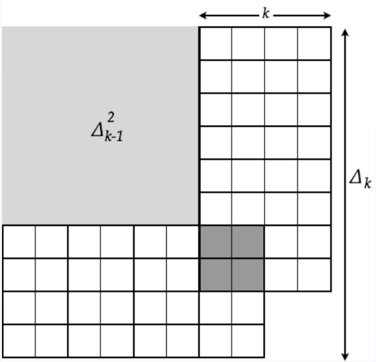


Because k is odd, the small squares pack perfectly since $(k-1)$ is even, hence divisible by 2.
The depicted case depicts pictorially the definition of triangular numbers: $k \cdot \frac{1}{2}(k-1) \;=\; \Delta_{k-1}$.

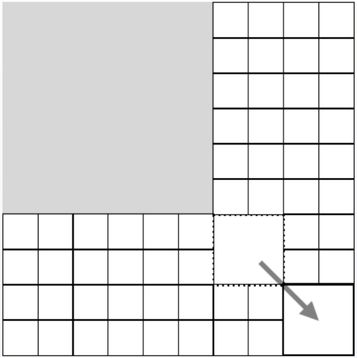
Inductive extension for even k.

The basic reasoning here mirrors that for odd k, with one small difference.

Now, as we assemble small squares around the large square, two subsquares overlap, as depicted below.



We must manipulate the overlapped region in order to get a tight packing around the large square.



Happily, when there is a small overlapping square region, there is also an identically shaped empty square region, as suggested by these two figures.

Conclusion.
Because $(k-2)$ is even, the like-configured square blocks can be allocated to two sides of the initial $\Delta_{k-1} \times \Delta_{k-1}$ square (namely, its right side and its bottom).
The overlap has the shape of a square that measures $\frac{1}{2}(\Delta_k - \Delta_{k-1})$ on a side.
One also sees in the figure an empty square in the extreme bottom right of the composite $\Delta_k \times \Delta_k$ square, which matches the overlapped square identically. This situation is the pictorial version of the equation

$$\Delta_k^2 \; - \; \Delta_{k-1}^2 \;\; = \;\; \frac{1}{4}k^2\left((k+1)^2 - (k-1)^2\right) \;\; = \;\; k^3$$

We have thus extended the inductive hypothesis for both odd and even k, whence the result. $\square$