Lecture 3 – Maths for Computer Science
Solving recurrences and Fibonacci numbers

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Lecture notes MoSIG1

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Various problems

Fibonacci sequence

$F(n+1) = F(n) + F(n-1)$ with $F(0) = 1$ and $F(1) = 1$

Lucas' numbers

Same as Fibonacci with a different seed.
$L(n+1) = L(n) + L(n-1)$ with $L(0) = 1$ and $L(1) = 3$

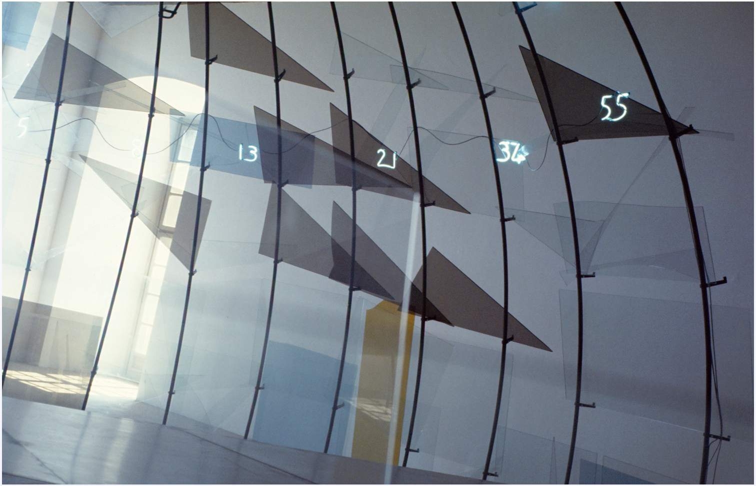
Derangements

$d(n+1) = n\,(d(n-1) + d(n-2))$ with $d(0) = 1$ and $d(1) = 2$

Stern sequence

$s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$ with $d(0) = 0$ and $d(1) = 1$

Fibonacci numbers are everywhere



Definition of Fibonacci numbers

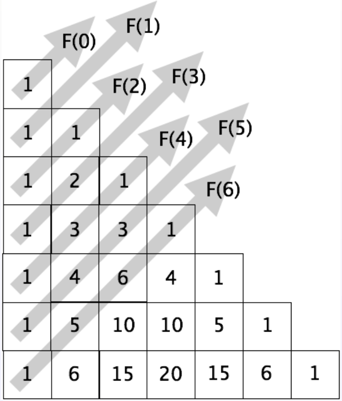
- Fibonacci numbers are the number of pairs of rabbits that can be produced at the successive generations.
- Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition:

Given the two numbers $F(0) = 1$ and $F(1) = 1$
the Fibonacci numbers are obtained by the following expression:
$F(n+1) = F(n) + F(n-1)$

Link with the Pascal's triangle

Could you prove the following property?



Cassini's identity

Proposition:

$F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1}$ for $n \geq 1$

Let check the expression on the first ranks:

$n = 1,\ F(0).F(2) = F(1)^2 + 1 = 2$

$n = 2,\ F(1).F(3) = F(2)^2 - 1 = 3$

$n = 3,\ F(2).F(4) = F(3)^2 + 1 = 10$

$n = 4,\ F(3).F(5) = F(4)^2 - 1 = 24$

...

Proof (by induction)

- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini's identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n)^2+F(n).F(n+1)$

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  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n)^2+F(n).F(n+1)$
  Replace the last term using the recurrence hypothesis:
  $F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1}$
  $= F(n-1).F(n+1) + (-1)^{n+2}$

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  Thus,
  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$
  $= F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$

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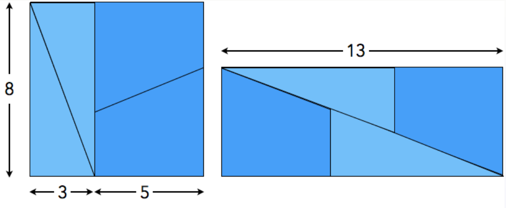
  $= F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$
  Apply again the definition of Fibonacci sequence
  $F(n) + F(n-1) = F(n+1)$, we obtain:
  $F(n).F(n+2) = F(n+1)^2 + (-1)^{n+2}$

A Paradox (favorite puzzle of Lewis Carroll)

Consider a chess board (8 by 8 square) and cut it into 4 pieces, then reassemble them into a rectangle.



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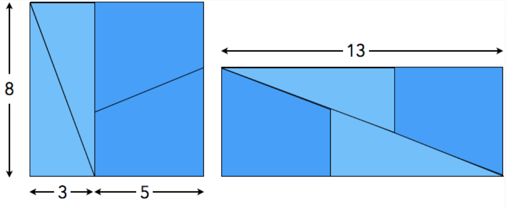


The surface of the square is $F(n)^2$ while the rectangle is $F(n+1).F(n-1)$.

The Cassini identity is applied for $n = 5$, $F(5) = 8$.

- On one side, the surface is $8 \times 8 = 64$
- On the other side $13 \times 5 = 65$

What's wrong?

Explanation

The paradox comes from the representation of the "diagonal" of the rectangle which does not coincide with the hypothenuse of the right triangles of sides $F(n+1)$ and $F(n-1)$.
In other words, it always remains (for any n) an empty space (corresponding to the unit size of the basic square of the chess board).

The greater n, the better the paradox because the *deformation* of the surface of this basic square becomes more tiny.

Computing $F(n)$ fast

$F(n)$ can be computed in $log_2(n)$ steps.

Proposition.

For all integers n:
(a) $F(2n) \;=\; (F(n))^2 \;+\; (F(n-1))^2$
(b) $F(2n+1) \;=\; F(n) \times (2F(n-1) + F(n))$

Details (a) – Proof by induction

The base case $n = 1$ is true because

$$\begin{aligned} F(2) &= (F(1))^2 + (F(0))^2 = 2 \\ F(3) &= F(1) \times (2F(0) + F(1)) = 3 \end{aligned}$$

Assume that the property holds for n, for both $F(2n)$ and $F(2n + 1)$.

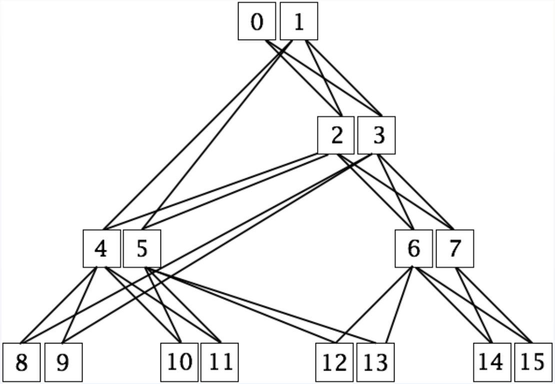
$$F(2(n+1)) = F(2n+1) + F(2n)$$
$$= (F(n))^2 + (F(n-1))^2 + F(n) \times (2F(n-1) + F(n))$$
$$= (F(n))^2 + (F(n-1))^2 + 2(F(n) \times F(n-1)) + (F(n))^2$$
$$= (F(n) + F(n-1))^2 + (F(n))^2$$
$$= (F(n+1))^2 + (F(n))^2$$

Details (b)

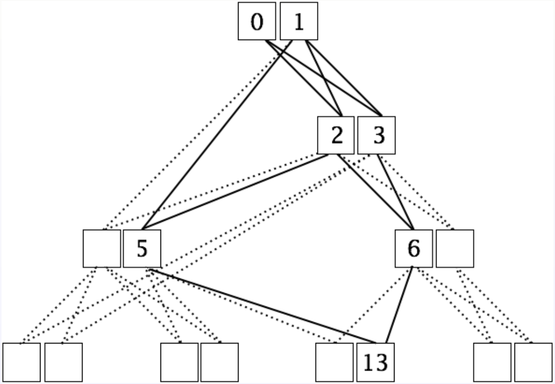
We again start by applying the defining recurrence of the Fibonacci numbers on $F(2(n+1)+1)$

$$= F(2(n+1)) \; + \; F(2n+1)$$
$$= (F(n+1))^2 \; + \; F(n)^2 \; + \; F(n) \times (2F(n-1) + F(n))$$
$$= (F(n+1))^2 \; + \; 2(F(n-1) + F(n)) \times F(n)$$
$$= (F(n+1))^2 \; + \; 2F(n+1) \times F(n)$$

Pictorially



Pictorially (from one node)



Definition of Lucas' numbers

A natural question is:

what happens if we change the first ranks of the sequence keeping the same recurrence pattern?

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It has been studied by the french mathematician Edouard Lucas, starting at 2 and 1 .

For some reasons that will be clarified later, the sequence is shifted backwards (we take the convention $L(-1) = 2$).

Definition of Lucas' numbers

Definition:
Given the two numbers $L(0) = 1$ and $L(1) = 3$
all the other Lucas' numbers are obtained by the same progression as Fibonacci:

- $L(n+1) = L(n) + L(n-1)$

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all the other Lucas' numbers are obtained by the same progression as Fibonacci:

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n	-1	0	1	2	3	4	5	6	7	8	9	10
F(n)		1	1	2	3	5	8	13	21	34	55	...
L(n)	2	1	3	4	7	11	18	29	47	76	123	...

- There are[1] strong links with Fibonacci numbers.

In particular, we established before that:

$F(n+2) = 1 + \sum_{k=0}^{n} F(k)$.

We have similarly:

$L(n+2) = 1 + \sum_{k=-1}^{n} L(k)$
since the basic step of the induction is still valid[2].
$L(2) = L(-1) + L(0) + 1 = 2 + 1 + 1 = 4$.

[1] of course
[2] It will be true for all the progressions where $u_1 = 1$

A first Property

We can also easily show that the Lucas number of order n is the sum of two Fibonacci numbers:

Proposition.

$L(n) = F(n-1) + F(n+1)$ for $n \geq 1$

Let *check* this property on the first ranks:

$n = 2,\ L(2) = F(1) + F(3) = 1 + 3 = 4$

$n = 3,\ L(3) = F(2) + F(4) = 2 + 5 = 7$

$n = 4,\ L(4) = F(3) + F(5) = 3 + 8 = 11$

$n = 5,\ L(5) = F(4) + F(6) = 5 + 13 = 18$

...

Proof by induction

- The **basis case** (for $n = 1$) is true since
  $L(1) = 3 = F(2) + F(0) = 2 + 1.$
- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n + 1)$:
  Apply the definition of Lucas' numbers:
  $L(n + 1) = L(n) + L(n - 1)$

Proof by induction

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- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$

Proof by induction

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  Apply the definition of Lucas' numbers:
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  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$

Proof by induction

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- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$ replace them in the previous expression:
  $L(n+1) = F(n+2) + F(n)$

which concludes the proof.

Extension 1

Notice that using a similar approach, we obtain
$L(n) = F(n+2) - F(n-2)$.

What happens if we generalize?

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Proposition.

$2.L(n) = F(n+3) + F(n-3)$

Proof.

We start from $L(n) = F(n+2) - F(n-2)$
$F(n+2) = F(n+3) - F(n+1)$ and
$F(n-2) = F(n-1) - F(n-3)$
$L(n) = F(n+3) - (F(n+1) + F(n-1)) + F(n-3)$
$2.L(n) = F(n+3) + F(n-3)$

Extension 2

Go to the next step using the same technique:

$$2.L(n) = F(n+3) + F(n-3)$$
$$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$$

$$3.L(n) = F(n+4) - F(n-4)$$

[3] The formal proof is let to the reader

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One more step: $5.L(n) = F(n+5) + F(n-5)$

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$3.L(n) = F(n+4) - F(n-4)$

One more step: $5.L(n) = F(n+5) + F(n-5)$

Thus, we guess the following expression.

Proposition[3].
$F(k-1).L(n) = F(n+k) + (-1)^{k-1}F(n-k)$ for $k \leq n$

[3] The formal proof is let to the reader

Two other propositions

Proposition.

$F(n+1) = \frac{1}{2}(F(1).L(n) + F(n).L(1))$

The proof comes from direct arithmetic manipulations:

$2.F(n+1) = F(n+1) + F(n+1)$

$= F(n+1) + F(n) + F(n-1)$

$= L(n) + F(n)$

$= F(1).L(n) + F(n).L(1)$

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$= L(n) + F(n)$

$= F(1).L(n) + F(n).L(1)$

The previous property can be extended for any $k > 1$
Let compute the expression of $F(k).L(n) + F(n).L(k)$

A final natural question

The golden ratio.

It is a well-known result that the ratio of two consecutive Fibonacci number tends to the Golden ratio:
$lim_{n \to \infty} \frac{F(n)}{F(n-1)} = \Phi$

As this result is obtained by solving the following equation $x^2 = x + 1$ (Φ is the positive root) and does not depend on the first rank, this holds also for the Lucas' numbers.

A last result: the Zeckendorf's Theorem

Objective: Study the Fibonacci numbers as a numbering system.

Let us first introduce a notation: $j \gg k$ iff $j \geq k + 2$.
The *Zeckendorf's theorem* states that:

every positive integer n has a unique decomposition of the form:
$n = F_{k_1} + F_{k_2} + ... + F_{k_r}$ where $k_1 \gg k_2 \gg ... \gg k_r$ and $k_r \geq 2$

Here, we assume that the Fibonacci sequence starts at index 1 and not 0, moreover, the decompositions will never consider $F(1)$ (since $F(1) = F(2)$).

Derangements

Derangements represent one of the simplest forms of *avoidance problems*.

- A professor views it as a win-win strategy for the students in her class to grade each others' essays on *The Essential Truth in the Universe*.
  The essays thereby get graded faster.
- Moreover, each student gets a chance to see how another student has interpreted some basic component of the human experience.
- The only complication is: How should we allocate essays among the students?

The process must ensure that no student is assigned her own essay to critique.

This challenge is known as a *derangement problem*.

Derangements

- A *derangement* of a (finite) set A is a *bijection* $f : A \leftrightarrow A$ that has no *fixed point*.

In other words, for every $a \in A$, we must have $f(a) \neq a$.

Clearly, derangements always exist (for $n > 1$).

One can just label the elements of set A by the numbers $0, 1, \ldots, |A| - 1$ and specify $f(a) = a + 1 \bmod |A|$.

Playing with a simple example

However, derangements are not so common! In fact, the set $A = \{0, 1, 2\}$ admits six self-bijections, but only two are derangements. Which ones?

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However, derangements are not so common! In fact, the set $A = \{0, 1, 2\}$ admits six self-bijections, but only two are derangements. Which ones?

$$f(a) = a + 1 \bmod 3 \quad : \text{which maps} \quad (0 \to 1), (1 \to 2), (2 \to 0)$$

$$g(a) = a - 1 \bmod 3 \quad : \text{which maps} \quad (0 \to 2), (1 \to 0), (2 \to 1)$$

- How many derangements does an arbitrary n-element set A have? We denote this quantity by $d(n)$.

Derangements

We compute $d(n)$ for arbitrary integer n via the following recursion:

- For $n = 1$: $d(1) = 0$.

  The unique bijection in this case consists only of a fixed point.
- For $n = 2$: $d(2) = 1$.

  There are two bijections in this case
  - the identity, which has two fixed points
  - the swap, which is a derangement.

The inductive expression

- For $n > 2$: $d(n) = (n-1)(d(n-1) + d(n-2))$:

  To see this, note first that in any derangement, the first element of A, call it a, must map to some $b \neq a$.

  - Note next that there are $n-1$ ways to choose b.
  - There are $d(n-2)$ derangements under which b maps to a. In those cases, we know everything about a and b, so we need worry only about the remaining elements of A.
    These $n-2$ elements can "derange" in all possible ways.
  - There are $d(n-1)$ derangements under which element b does not map to a.

An observation

The preceding reasoning verifies the following recurrence

$$d(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ n(d(n-1) + d(n-2)) & \text{if } n > 2 \end{cases}$$

Solving the recurrence

There are several ways to solve this recurrence.

- We can reduce the bilinearity by a linear recurrence:

$$d(n) = \begin{cases} 0 & \text{if } n = 1 \\ n.d(n-1) + (-1)^n & \text{if } n > 1 \end{cases}$$

Interestingly, as the number of objects in the set to be deranged grows without bound.

The proportion of bijections that are derangements tends to the limit $1/e$, where e is the base of natural logarithms.

Stern's sequence

Definition

$s(0) = 0$ and $s(1) = 1$

$s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$

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Interpretation:

- If n is even, we keep the value $s(n/2)$
- If it is odd, we split it into two parts that are as balanced as possible.

Get an insight

First elements

1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1 6

Get an insight

First elements

1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1	6
1	**1**	2	**1**	3	2	3	**1**	4	3	5	2	5	3	4	**1**	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	**1**	6

- What is the *best* representation?

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

Sum of the elements in a row

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

Sum of the elements in a row

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

- It is a power of 3.

progression of the elements along a given column

- It is an arithmetic progression.
- Consider $s(n)$ as $s(p, q)$, for $p \geq 0$ and $1 \leq q \leq 2^p$

p \ q																							
1																							
1	2																						
1	3	2	3																				
1	4	3	5	2	5	3	4																
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11

Another representation

1 1 **2** 1 3 **2** 3 1 4 3 5 **2** 5 3 4 1 5 4 7 3 8 5 7 **2** 7 5 8 3 7 4 5

Another representation

1	1	**2**	1	3	**2**	3	1	4	3	5	**2**	5	3	4	1	5	4	7	3	8	5	7	**2**	7	5	8	3	7	4	5

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

Terms in a row

- They are arranged in a symmetric order and more precisely, like a palindrome.

Maximum number in each row

1																							
1	**2**																						
1	**3**	2	3																				
1	4	3	**5**	2	5	3	4																
1	5	4	7	3	**8**	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	**13**	5	12	7	9	2	9	7	12	5	13	8	11

Maximum number in each row

1																							
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1	4	3	**5**	2	5	3	4																
1	5	4	7	3	**8**	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	**13**	5	12	7	9	2	9	7	12	5	13	8	11

- They are the successive Fibonacci numbers

More properties

- Deriving the rationals
- Combinatorial proof
- Links with the Pascal's triangle