Maths for Computer Science
Summations

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Introduction

Illustration of methodological element

We investigate here a useful mathematical technique.

- Stand alone toolbox.
- As an inspiring element.
- No need to rely on sophisticated material.

Computing Geometric series

let n be an integer, $\Sigma_{k=0,n}2^k = ?$

This is a particular case ($a = 2$) of the **geometric progression**.
$S_a(n) = \Sigma_{k=0,n}a^k = \frac{a^{n+1}-1}{a-1}$ for $a \neq 1$

Let us expand the summation:

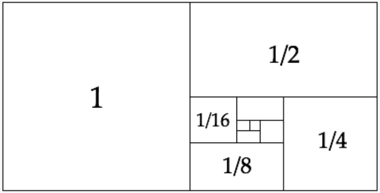
- $S_a(n) = 1 + a + a^2 + \cdots + a^n$
- $= 1 + a[1 + a + a^2 + \cdots + a^{n-1}] + a^{n+1} - a^{n+1}$
- $= 1 + a \cdot S_a(n) - a^{n+1}$
- Thus, $(1 - a)S_a(n) = 1 - a^{n+1}$

Remark that most existing proofs directly suggest to multiply $S(n)$ by $1 - a$

Other ways of computing particular geometric series

$a = \frac{1}{2}$

Using an analogy with geometry (surface of the unit square).



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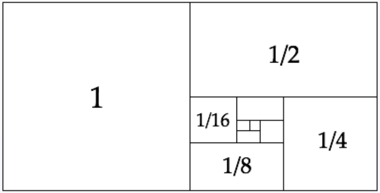
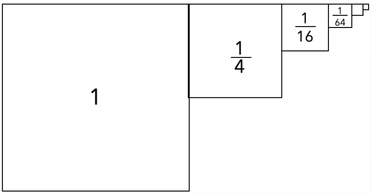


Remark: We may also have used unit sized disks...

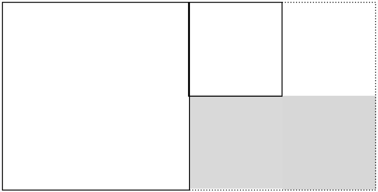
Particular geometric series

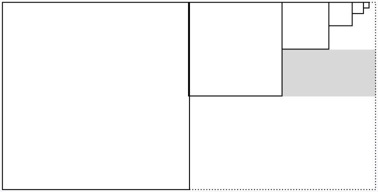
$a = \frac{1}{4}$

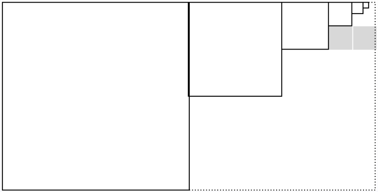
Could we generalize the previous construction?

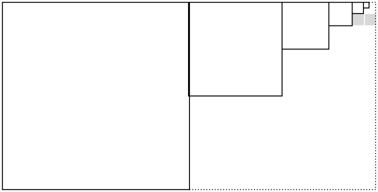


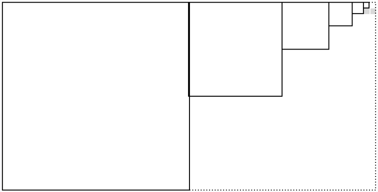
- Similarly as before, the area of the 1 by 1 square is 1.
- Let us determine the whole surface.

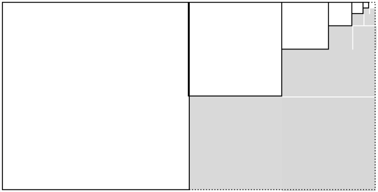








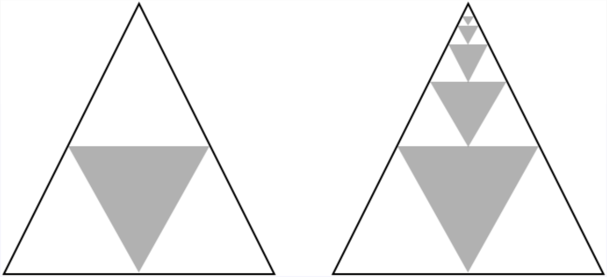




- The grey area is twice the area on the square at the right.
- Thus, the whole area is $1 + \frac{1}{3}$

Another construction

A pattern easier to analyze for $a = \frac{1}{4}$



- Assuming the base triangle area is 1, the solution is the grey area.
- Argument: It is one third at each layer.

Exercise

Prove this result formally.

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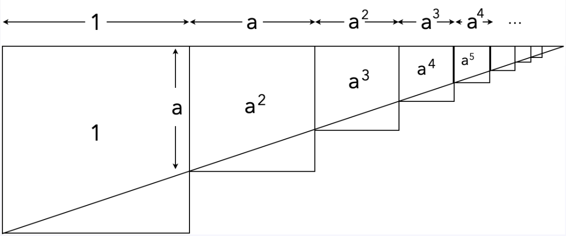
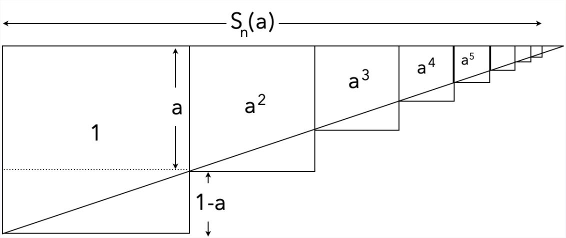
$S_{1/4} = \frac{1}{3} + 1 = \frac{4}{3}$

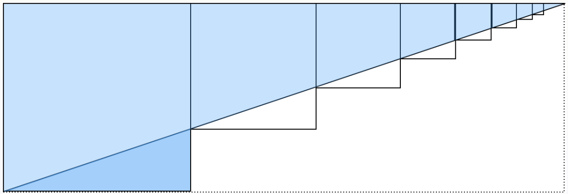
- What happens at infinity?
- Are you sure we told the whole story in a proper way?

Generalization:
Any geometric series with $a < 1$




$S_n(a)$
1
a
a^2
a^3
a^4
a^5
1-a

Analysis



- The value of the summation is given by the Thales' theorem (triangle similarity)[1]
- The theorem states that the ratios of the sides of similar right triangles remain the same
- Thus, $\frac{S_n(a)}{1} = \frac{1}{1-a}$

[1] notice here the transversality of the topics in Maths

Identities

$$a^n - b^n = ?$$

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$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + b^{n-1}).$

The proof technique works exactly as before by cancelling pairs of equal terms!

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$(a+b)^n = ?$
The second one is the classical Newton binomial expression.

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Harmonic series

What are the values of $\Sigma_{k\geq 0}\frac{1}{2^k}$ and $\Sigma_{k>0}\frac{1}{k}$?

$\Sigma_{k>0}f(k) = lim_{n\to\infty}\Sigma_{k=1,n}f(k)$
Obtaining a finite value for an infinite sum was a paradox for a long time until the infinitesimal calculus of Leibniz/Newton on the XVIIth century.

- The *limit* of the first sum is 2.
  This is obtained by using the sum of a geometric progression for $a = \frac{1}{2}$.
- The second sum is unbounded (it goes to $+\infty$).
  The result is obtained by bounding the summation:
  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... > 1 + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} + ...$
  and the infinite sum of positive constant numbers (here $\frac{1}{2}$) is infinite.

An extra (related) question

Compute extended geometric series and their sums

$S_a^{(c)}(n) = \sum_{i=1}^{n} \ i^c a^i$

where c is an arbitrary fixed positive integer, and a is an arbitrary fixed real number.

We restrict attention to summations $S_a^{(c)}(n)$ that satisfy the joint inequalities $c \neq 0$.

- We have already adequately studied the case $c = 0$, which characterizes "ordinary" geometric summations.
- Assume and $a \neq 1$ since the degenerate case $a = 1$ removes the "geometric growth" of the sequence underlying the summation.

Summation method

- The method is *inductive in parameter c*, for each fixed value of c, the method is *inductive in the argument n*.
  We restrict our study to the case $c = 1$.

The summation $S_a^{(1)}(n) = \sum_{i=1}^{n} ia^i$

Proposition.

For all bases $a > 1$,

$$S_a^{(1)}(n) \;=\; \sum_{i=1}^{n} ia^i \;=\; \frac{(a-1)n-1}{(a-1)^2} \cdot a^{n+1} \;+\; \frac{a}{(a-1)^2} \quad (1)$$

Proof

We begin to develop our strategy by writing the natural expression for $S_a^{(1)}(n) = a + 2a^2 + 3a^3 + \cdots + na^n$ in two different ways.

First, we isolate the summation's last term:

$$S_a^{(1)}(n+1) = S_a^{(1)}(n) + (n+1)a^{n+1} \tag{2}$$

Then we isolate the left-hand side of expression:

$$\begin{aligned} S_a^{(1)}(n+1) &= a + \sum_{i=2}^{n+1} ia^i \\ &= a + \sum_{i=1}^{n} (i+1)a^{i+1} \\ &= a + a \cdot \sum_{i=1}^{n} (i+1)a^i \end{aligned}$$

Proof

Let develop the last sum[2]:

$$\begin{aligned} &= a + a \cdot \left(\sum_{i=1}^{n} ia^i + \sum_{i=1}^{n} a^i \right) \\ &= a \cdot \left(S_a^{(1)}(n) + S_a^{(0)}(n) \right) + a \\ &= a \cdot \left(S_a^{(1)}(n) + \frac{a^{n+1} - 1}{a - 1} - 1 \right) + a \\ &= a \cdot S_a^{(1)}(n) + a \cdot \frac{a^{n+1} - 1}{a - 1} \qquad (3) \end{aligned}$$

[2] Could you guess "why?"

We now use standard algebraic manipulations to derive the expression

Combining both previous expressions of $S_a^{(1)}(n+1)$, we finally find that

$$\begin{aligned}(a-1)\cdot S_a^{(1)}(n) &= (n+1)\cdot a^{n+1} - a\cdot\frac{a^{n+1}-1}{a-1} \\ &= \left(n-\frac{1}{a-1}\right)\cdot a^{n+1} + \frac{a}{a-1} \qquad (4)\end{aligned}$$

Good exercise: check the previous calculations.

Another way to solve

Solving the case $a = 2$ using subsum rearrangement.

We evaluate the sum $S_2^{(1)}(n) \;=\; \sum_{i=1}^{n} \; i2^i$
in an especially interesting way, by rearranging the sub-summations of the target summation.

Underlying our evaluation of $S_2^{(1)}(n)$ is the fact that we can rewrite the summation as a *double* summation:

$$S_2^{(1)}(n) \;=\; \sum_{i=1}^{n} \sum_{k=1}^{i} 2^i \qquad (5)$$

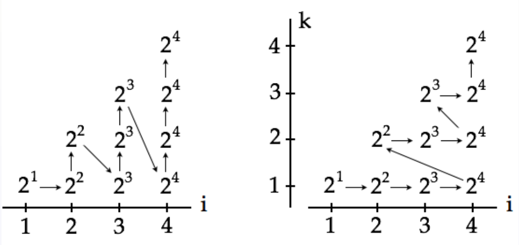
By suitably applying the laws of arithmetic specifically, the distributive, associative, and commutative laws, we can perform the required double summation in a different order than that specified previously.

We can exchange the indices of summation in a manner that enables us to compute $S_2^{(1)}(n)$ in the order implied by the following expression:

$$S_2^{(1)}(n) \;=\; \sum_{k=1}^{n} \sum_{i=k}^{n} 2^i$$

Are you able to validate this transformation?

An easier way to see the transformation

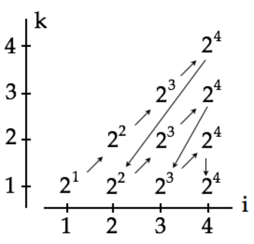


The indicated summation is much easier to perform in this order, because its core consists of instances of the "ordinary" geometric summation $\sum_{i=k}^{n} 2^i$.

Expanding these instances, we find finally that

$$\begin{aligned} S_2^{(1)}(n) &= \sum_{k=1}^{n} \left(2^{n+1} - 1 - \sum_{i=0}^{k-1} 2^i\right) \\ &= \sum_{k=1}^{n} \left(2^{n+1} - 2^k\right) \\ &= n \cdot 2^{n+1} \; - (2^{n+1} - 1) + 1 \\ &= (n-1) \cdot 2^{n+1} + 2 \end{aligned}$$

We remark that the process of obtaining the original summation can also be seen by scanning the elements of the summation along diagonals.



Each of the n diagonals contains exactly the difference between the complete geometric summation and the partial summation that is truncated at the kth term.

Final message

- We had a brief overview of techniques for manipulating the mathematical object of summation.
- We also started to write proper proofs.