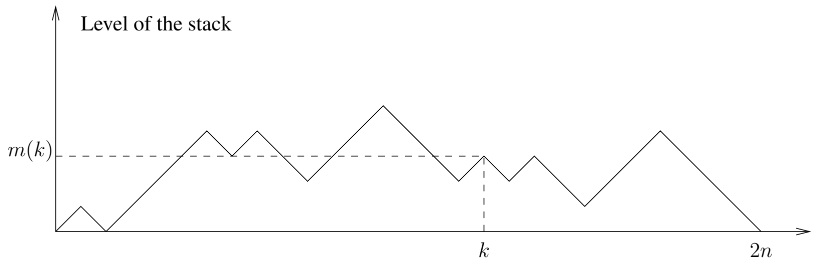
UE Mathematics for Computer Science

I. Counting Stack Orderings

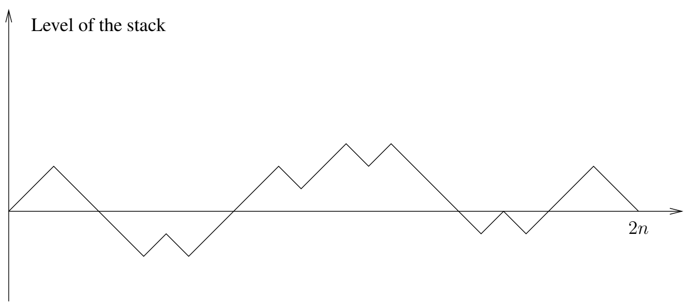
Consider a stack with the two primitives *push* and *pop*. An execution of a program consists in n operations *push* and n *pop* which could be interleaved. The execution is represented by a *mountain*, the function $m(k)$ that gives the level of the stack after k operations.



Denote by M_n the number of *mountains* with n *push* (up-stroke) and n *pop* (down-stroke) operations and set $M_0 = 1$.

I.1. **Small n cases**

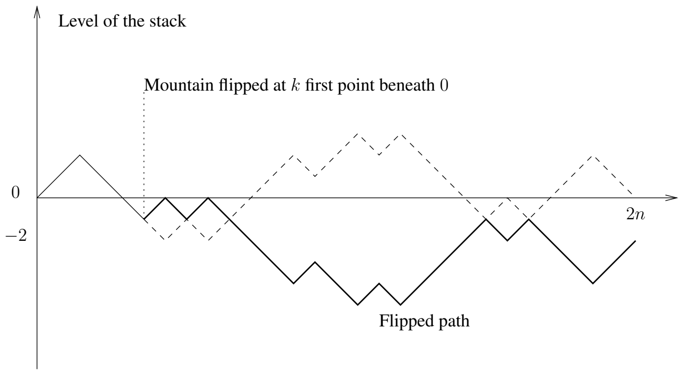
For $n = 1, 2, 3$ give the possible *mountains* and deduce M_1, M_2, M_3. An extended *mountain* of length $2n$ with n up-strockes and n down-strockes allows to be under the sea level (bad mountains):



I.2. **Extended *mountains***

Compute the number of extended mountains with length $2n$.

The flip operation consists in exchanging all the slopes after the first passage below 0:



I.3. **Flipped mountains**

Show that the set of bad *mountains* is in bijection with the set of *mountains* with $n-1$ up-strokes and $n+1$ down-strokes.

I.4. **Computation**

Prove that

$$M_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{(n+1)!n!}. \tag{1}$$

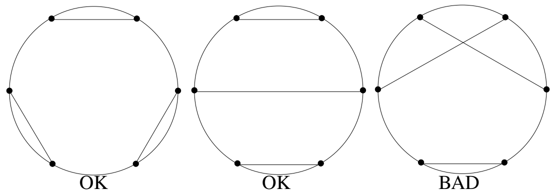
I.5. **Recurrence relation** Show directly on mountain diagrams that the M_n numbers satisfy the recurrence equation:

$$M_n = M_0 M_{n-1} + M_1 M_{n-2} + \cdots + M_{n-1} M_0. \tag{2}$$

II. Examples

II.1. **Shake hands**

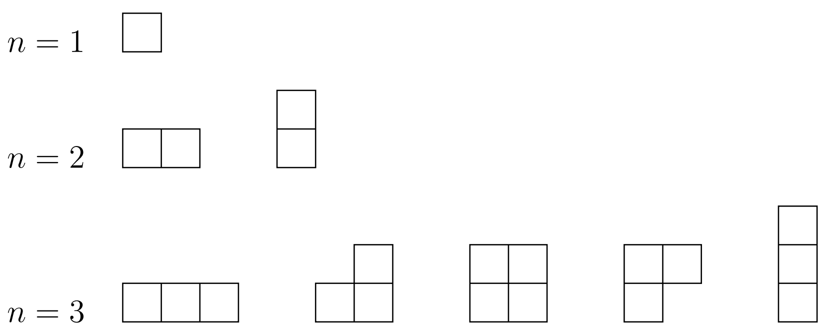
Some shaking hands configurations with $n = 3$



Suppose that $2n$ persons are seated around a table, how many ways could they shake hands without crossing ?

II.2. **Circuits**

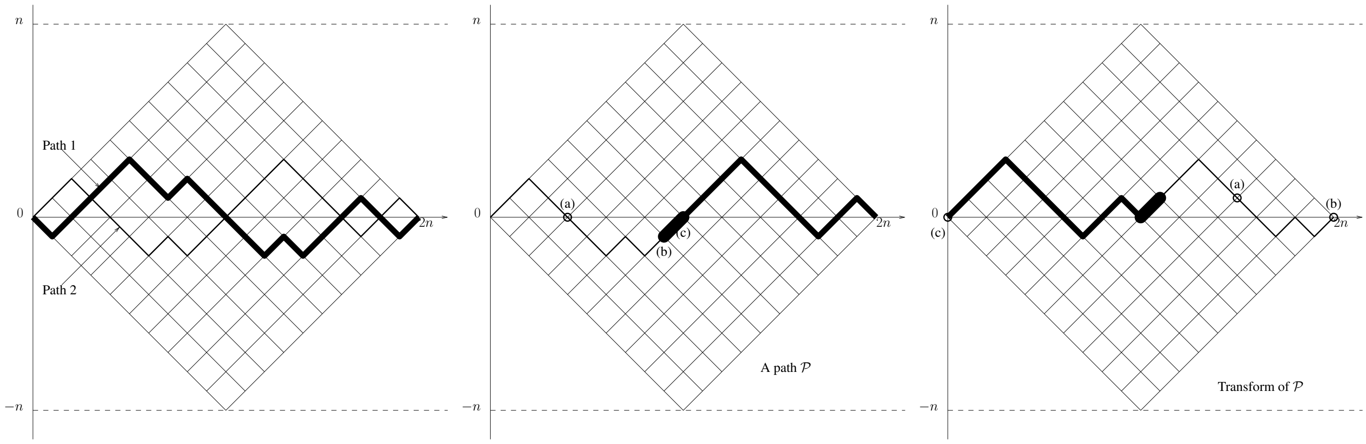
Circuits with perimeter $2n+2$



How many shapes of circuits could be done with $2n+2$ unit segments ?

III. Paths in a diamond *Exam 2018*

Consider paths of length $2n$ in a diamond of size $n \times n$ from $(0,0)$ to $(2n,0)$. Two paths are given in the Figure 1a.



(a) Two path in a diamond (b) A path $\mathcal{P}$ (c) Transform of path $\mathcal{P}$

Figure 1: Paths in a 10×10 diamond

III.1. Give all such paths for the 3×3 diamond.

III.2. Compute P_n the number of paths in a $n \times n$ diamond (give the proof).

For a path $\mathcal{P}$ we define the below number $B(\mathcal{P})$ counting the number of steps that are below the x-axis and going down.

III.3. What are the possible values of $B(\mathcal{P})$? Compute $B(\mathcal{P})$ for all the paths in the 3×3 diamond and for the two paths given in the Figure 1a.

Consider now the following transformation T of a path illustrated in the figures 1b and 1c. Take the first point (a) when the path cuts the x-axis, take the point (c) when the path hits again the x-axis for the first time by the step $(b) \to (c)$. Then rearrange the path by shifting the last part of the path up to 0, put the step $(b) \to (c)$ and finish by the first part of the path. Of course this transform applies on paths with positive below number

III.4. Prove and compute the inverse transform.

III.5. Prove that, if $B(\mathcal{P}) \geqslant 1$ we have $B(T(\mathcal{P})) = B(\mathcal{P}) - 1$. And deduce that the set of paths with below number p is in bijection with the set of paths with below number $p-1$.

III.6. What are the paths satisfying $B(\mathcal{P}) = 0$? How many are they ?

III.7. (optional) What is the name of the number of paths such that $B(\mathcal{P}) = 0$?

III.8. Prove that the set of paths that never go below the x-axis is in bijection with the set of complete[1] binary trees.

III.9. Propose an algorithm that generate uniformly a complete binary tree and compute its complexity.

[1] Complete means that each node has either 0 or 2 child.

IV. Miscellaneous Exercises

IV.1. **Choosing a team**

You want to choose a team of m people from a pool of n people for your startup company, and from these people you want to choose k to be the team managers. You took the *Mathematics for Computer Science course*, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your manager, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}$$

Before doing the reasonable thing, dump on your manager, you decide to check his answer against yours.

1. Start by giving an algebraic proof that your manager's formula agrees with yours.

2. Now give a combinatorial argument proving this same fact.

IV.2. **A curious decomposition**

Now try the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$$

1. Start with a combinatorial argument. Hint: let $\mathcal{S}$ be the set of all sequences in $\{0, 1, \star\}^n$ containing exactly one $\star$.
2. How would you prove it algebraically?

IV.3. **Covering**

Let $\mathcal{E}$ a set of n elements. A 2-covering is a couple subsets (A, B) of $\mathcal{E}$ such that $A \cup B = \mathcal{E}$. Compute the number of 2-covering.

IV.4. **No adjacency**

There are 20 books arranged in a row on a shelf.

1. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15bit sequences with exactly 6 ones.
2. How many ways are there to select 6 books so that no two adjacent books are selected?

IV.5. **Combinatorial identity**

Prove the following theorem

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

1. using a combinatorial argument;
2. using induction.