UNIVERSITÉ Grenoble Alpes — Master of Science in Informatics at Grenoble — Grenoble INP ensimag

UE Mathematics for Computer Science

First session exam December 7, 2017 (3 hours)

> **Important information.**
> **Read this before anything else!**
>
> - Only personal hand-written manuscripts and lecture notes are allowed.
> - Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
> - Please write your answers to each part on separate sheets of papers (3 separate sheets).
> - The different problems are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
> - All problems are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
> - The number of points alloted to each question gives you an indication on the expected level of details and on the time you should spend answering.
> - Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
> - The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged.**

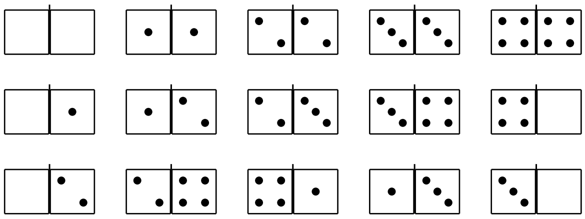
Indicative grades

	1 : Triominoes						2: Cake		3 : $\sqrt{2}$	4: Lucas' Number			5 : PN		
question	1	2	3	4	5	6	1	2	1	1	2	3 & 4	1	2	3
points	1	2	2	4	2	2	1	2	3	1	1	1	3	4.5	2

Part I : Tri-ominoes

Consider an extension of the domino game by using triangles instead of rectangles. Each corner of the triangle is marked by a number between 0 and n.

All domino tiles for $n = 4$



Some (not all) tri-omino tiles for $n = 5$

1. Compute the number T_n of tri-ominos with $n = 0, 1, 2, 3$.

   Hint : take care of rotations, the tri-omino $[1, 2, 3]$ *is the same as* $[2, 3, 1]$ *and* $[3, 1, 2]$ *but is different from* $[1, 3, 2]$.

2. Establish a recurrence equation on T_n and compute T_n

   Hint : take care of triple, double, etc.

We suppose given a function `RandomInt` $(0, n)$ that provides a sequence of random numbers, independent and uniformly distributed on $\{0, \cdots, n\}$

3. Propose an algorithm that generates random tri-ominoes, uniformly among all possible tri-ominoes of size n and prove it.

4. A rejection based method.

   (a) Why the following algorithm does not generate uniformly a domino ?

   Domino-GeneratorA(n)

   Data: n the maximum number on a domino
   Result: A random domino

   $i \leftarrow$ `RandomInt` $(0, n)$
   $j \leftarrow$ `RandomInt` $(0, n)$
   return $[i, j]$

   (b) Prove that the following algorithm provides uniform random donimoes.

   Domino-GeneratorB(n)

   Data: The maximum number on a domino
   Result: A random domino uniformly distributed

   repeat
   $\quad i \leftarrow$ `RandomInt` $(0, n)$
   $\quad j \leftarrow$ `RandomInt` $(0, n)$
   until $(i = j)$ *or* $((i \neq j)$ *and* `RandomInt`$(0, 1) = 1))$
   return $[i, j]$

(c) Adapt this approach to generate uniformly tri-ominoes.

In the real game, for the limitation of the number of triangles, the set of tri-ominoes is restricted to $[i, j, k]$ such that $i \leqslant j \leqslant k$

5. Compute the number of tri-ominoes $[i, j, k]$ such that $i \leqslant j \leqslant k$.
6. Generalize this result to compute the number of non-decreasing functions from $\{1, \cdots, k\}$ to $\{0, \cdots, n\}$. Propose an algorithm that generates uniformly such a function.

Part II : Some problems

2 Cake division

Some preliminaries: This problem has been studied in a training class, the cake is represented by the interval $[0, 1]$ associated with a *measure* $v_i(x)$ which represents how much agent i likes piece x. The measure is normalized, which means $\Sigma v_i([0, 1]) = 1\ \forall i$.

The moving-knife procedure is recalled below for n agents. There exists an external referee who is managing the knife.

i. The referee moves a knife slowly across the cake, from left to right. Any agent may shout "stop" at any time. Whoever does so receives the piece at the left of the knife.

ii. When a piece has been cut off, we continue with the remaining agents, until just one agent is left (who takes the rest).

1. Discuss briefly the protocol in term of proportional fairness, envy-free and equity (we can distinguish the cases for two or more agents).

Analysis of the *moving knife* protocol for 3 agents. The solution is described as follows with 4 cuts.

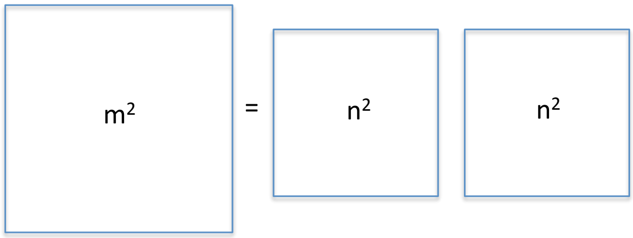
i. A referee slowly moves a knife across the cake, from left to right
ii. At the same time, each agent is moving his/her own knife so that it would cut the righthand piece in half (with regard to their own valuations).
iii. The first agent to shout "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which his/her knife is pointing.

2. Show the proportional fairness of this protocol. Then, prove that this protocol is envy-free.

3 Irrationality of $\sqrt{2}$

The proof is by contradiction, assuming $\sqrt{2}$ is rational (it can be written as the ratio of two integers $\frac{m}{n}$). Consider that the fraction can not be simplified and it is the smallest one.

Thus, $m^2 = 2n^2$, which may be represented geometrically as follows.



1. Show the contradiction by a geometrical proof (describe the principle and draw the corresponding figure).

4 Induction on Lucas' numbers

Recall the definition of Lucas' numbers which are an extension of Fibonacci numbers.

Given the two numbers $L_0 = 2$ and $L_1 = 1$, the Lucas numbers are obtained by the same progression as for Fibonacci: $L_{n+1} = L_n + L_{n-1}$.

1. Show that the Lucas number of order n is the average of the two previous and following Fibonacci numbers:

$$L_n = F_{n+1} + F_{n-1}.$$

2. Determine more generally the expression of L_n with $F_{n+k} + F_{n-k}$ for $1 \leqslant k \leqslant n$.

3. Show that

$$2L_{n+m} = F_m.L_n + F_n.L_m.$$

4. Show that

$$F_{2n} = L_n F_n.$$

Part III : Perfect Numbers

5 Perfect numbers

A perfect number (*PN* in short) is a number which is equal to the sum of its proper divisors. Example 28 whose 5 proper divisors are: 1, 2, 4, 7 and 14 since $1 + 2 + 4 + 7 + 14 = 28$.

We start by studying the numbers on the form: $2^\alpha - 1$

1. **Little Fermat Theorem, classic proof**
   (a) Show that $2^\alpha - 1$ is only prime if α is prime.
   Hint: The proof is based on the little Fermat theorem, which may be stated as follows:
   For all prime number p and for all integer α we have $\alpha^p - \alpha$ is divisible by p.
   (b) Show the classical proof by applying the Newton binomial decomposition.

2. **Little Fermat Theorem, combinatorial proof**
   Consider an alphabet $\mathcal{A}$ with a symbols.
   (a) What is the number of words of length p?
   (b) Show all words of length $p = 3$ on $\mathcal{A} = \{A, B\}$.

   We define the circular permutation c of a word as taking the last symbol of the word and putting it in the first position. c(ABCD) = DABC.
   (c) Give the graph of c for words of length $p = 3$ on $\mathcal{A} = \{A, B\}$.
   Hint: Define the necklace $\mathcal{N}(w)$ associated to a word w as the set of successive images of w by c. N(w) = w,c(w),c(c(w)),.... The size of a necklace is the number of elements in N(w).
   (d) Prove that the size of a necklace is always bounded by p. Compute the number of necklaces of size 1.
   (e) Prove that if p is prime the size of a necklace with at least 2 different symbols is p.
   (f) Conclude on the proof of Fermat's little theorem with combinatorial arguments.

3. **Properties of Perfect Numbers**
   α is a prime. Let denote by PN_α the number obtained by $2^{\alpha-1}(2^\alpha - 1)$ where $2^\alpha - 1$ is a prime. For instance, $\alpha = 3$ corresponds to $4 \times 7 = 28$.
   (a) Show that PN_α is a perfect number and all the perfect numbers have this form.
   (b) Show that the last digit of any perfect number (in usual decimal notation) is 6 or 8.
   (c) Show that $PN_\alpha = \Delta_{2^\alpha - 1}$.