UE Mathematics for Computer Science

First session exam December 13, 2016 (3 hours)
Only personal hand-written notes are allowed.
Use separated sheets for Part I and Part II. All problems are independent from each other.
Number of points given for each problem is given for information purposes only and is subject to modifications without notice.

Part I

Jerry and the cheese $\sim \frac{1}{2}$ *hour*

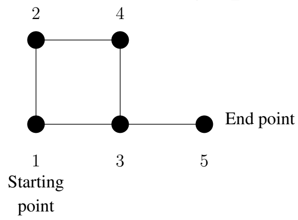
Consider a random walk on a non directed connected graph $\mathcal{G} = (\mathcal{X}, \mathcal{E})$, $\mathcal{X}$ is the set of nodes and $\mathcal{E}$ the set of edges. For x a node of the graph, $\mathcal{V}_x$ is the set of neighbors of x in $\mathcal{G}$.

Denote by X_n the position at step n of the random walk, X_0 is the initial position and the sequence $\{X_n\}_{n\in\mathbb{N}}$ is the trajectory (path) followed by the random walk. When the random walk is on some node x it jumps to the next node uniformly and independently from all other jumps to a neighbor y of x. So the transition probability is

$$p_{x,y} = \begin{cases} \frac{1}{|\mathcal{V}_x|} & \text{if } y \in \mathcal{V}_x; \\ 0 & \text{elsewhere.} \end{cases}$$

The aim of this exercise is to compute the average time needed by a random walk to reach some given vertex x_f starting from x_0 .

1. Consider the following example with 5 nodes graph

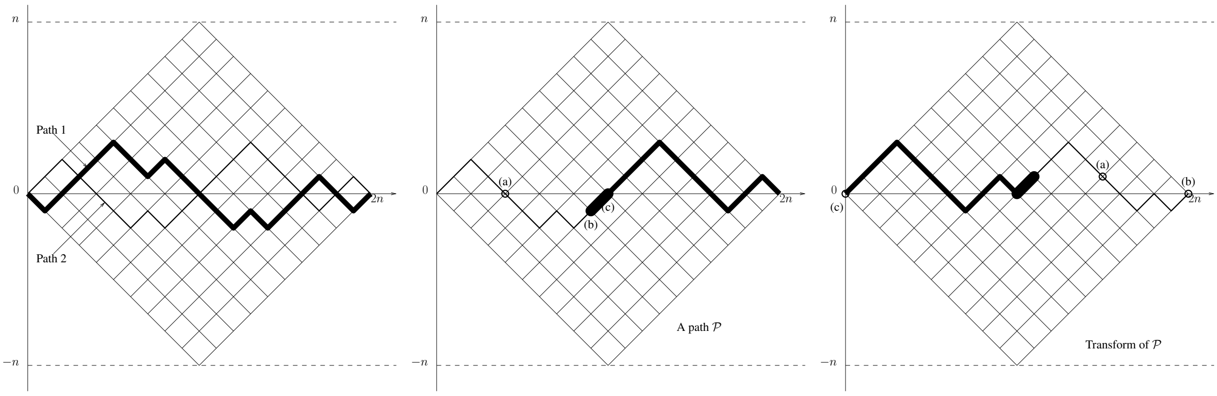


   (a) Explain briefly why the process $\{X_n\}_{n\mathbb{N}}$ is a homogeneous Markov chain and give the associated transition matrix.

   (b) The random walker is starting in node 1 and is randomly jumping from node to node. Compute $h_{1,5}$ the average hitting time of node 5 starting from 1.[1]

2. Propose a generalization of the approach to compute for given x, y, $h_{x,y}$ the average hitting time of y starting from x.

[1]Hint : compute the average hitting time from all possible starting nodesas a solution of a linear system



(a) Two path in a diamond (b) A path $\mathcal{P}$ (c) Transform of path $\mathcal{P}$

Figure 1: Paths in a 10×10 diamond

Paths in a diamond ~ 1 ***hour***

Consider paths of length $2n$ in a diamond of size $n \times n$ from $(0, 0)$ to $(2n, 0)$. Two paths are given in the Figure 1a.

1. Give all such paths for the 3×3 diamond.
2. Compute P_n the number of paths in a $n \times n$ diamond (give the proof).

For a path $\mathcal{P}$ we define the below number $B(\mathcal{P})$ counting the number of steps that are below the x-axis and going down.

3. What are the possible values of $B(\mathcal{P})$? Compute $B(\mathcal{P})$ for all the paths in the 3×3 diamond and for the two paths given in the Figure 1a.

Consider now the following transformation T of a path illustrated in the figures 1b and 1c. Take the first point (a) when the path cuts the x-axis, take the point (c) when the path hits again the x-axis for the first time by the step $(b) \to (c)$. Then rearrange the path by shifting the last part of the path up to 0, put the step $(b) \to (c)$ and finish by the first part of the path. Of course this transform applies on paths with positive below number

4. Prove and compute the inverse transform.
5. Prove that, if $B(\mathcal{P}) \geqslant 1$ we have $B(T(\mathcal{P})) = B(\mathcal{P}) - 1$. And deduce that the set of paths with below number p is in bijection with the set of paths with below number $p - 1$.
6. What are the paths satisfying $B(\mathcal{P}) = 0$? How many are they ?
7. (optional) What is the name of the number of paths such that $B(\mathcal{P}) = 0$?
8. Prove that the set of paths that never go below the x-axis is in bijection with the set of complete[2] binary trees.
9. Propose an algorithm that generate uniformly a complete binary tree and compute its complexity.

[2]Complete means that each node has either 0 or 2 child.

Part II

Handling numbers $\sim \frac{1}{4}$ *hour*

1. Show by a geometrical proof that the odd square numbers are congruent to 1 modulo 8.

   This is the case for $3^2 = 8 + 1$, $5^2 = 3 * 8 + 1$, $7^2 = 6 * 8 + 1$, etc..

2. Show by a geometrical proof that the product of any four consecutive integers is equal to a square minus 1.

   For instance: $3 * 4 * 5 * 6 = 19^2 - 1$

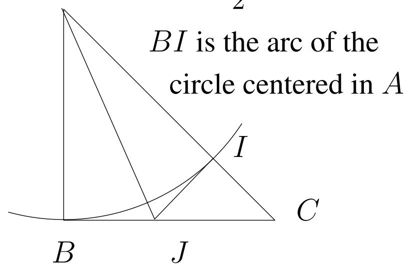
   Hint: Compare the extreme product $n(n + 3)$ *to the medium one* $(n + 1)(n + 2)$.

Some applied Graph Theory $\sim \frac{1}{4}$ *hour*

7 students have been elected to the academic council of the new Grenoble-Alpes university. They should vote for the construction of a new sport hall or a new lecture hall. The vote required the absolute majority. All these students have an opinion (this means that they do not hesitate between two options), however, before the final vote, they meet two by two for exchanging about their opinion. Each one has the time to meet only 3 other persons.

1. Show that whatever the opinion is, it is impossible to predict the issue of the vote.

Irrational numbers $\sim \frac{1}{2}$ *hour*



Let first recall that an *irrational number* is any real number that cannot be written as a fraction of two integers. Intuitively, this means that such numbers can not be represented by a simple fraction. They are those numbers that have an infinite number of digits in the decimal notation like π.

Prove that $\sqrt{2}$ is irrational by two methods.

1. Prove the result by contradiction. *(Hint: it uses simple parity arguments*

   Thus, assume that $\sqrt{2} = \frac{a}{b}$ where a and b are relatively primes ($GCD(a, b) = 1$).

2. Prove this result by a purely geometric method.

   Let consider a right triangle whose two sides are equal (such a triangle is said *isosceles*) and assume this side is integer. Using the well-known Pythagorean theorem, show first that $\sqrt{2}$ is rational if and only if the hypothenuse is an integer.

   Assume now by contradiction that this is the isosceles triangle (A, B, C) with the smallest area and construct another one with a lower surface.

Card tricks $\sim \frac{1}{2}$ ***hour***

The principle is as follows: someone draws a card from the deck. He/she puts it in some place into the deck. The cards are shuffled (several times) according to the same given permutation.

The following trick is based on the *Fermat's little theorem*, which states that if p is a prime number and α is an integer that is not divisible by p then $\alpha^{p-1} - 1$ is a multiple of p.

Using a more formal notation: $\alpha^{p-1} \equiv 1\ [p]$.

Let us start by proving this theorem by induction, inspired by the elegant solution proposed by Leonard Euler. Two integers are concerned here (α and p), but since the distribution of primes is irregular and not precisely known, it is natural to make an induction over α with fixed p.

1. Check the basis of the induction
2. **Induction step:**

   Let consider a prime p. We want to prove that the property holds for $\alpha + 1$ which is relative prime of p, assuming $\alpha^{p-1} - 1$ is a multiple of p.

   Show that $(\alpha+1)^p - (\alpha^p + 1)$ is a multiple of p using the *Newton decomposition* on $\alpha + 1$.
3. Use the induction hypothesis to know that $\alpha.(\alpha^{p-1} - 1)$ is proportional to p.
4. Conclude $(\alpha+1)^{p-1} \equiv 1\ [p]$.

In-shuffles We are given $2n$ cards, which are divided evenly into two decks. The in-shuffle corresponds to put them such that the cards in the final deck come alternatively from left to right as depicted in Figure 2 for $n = 4$.

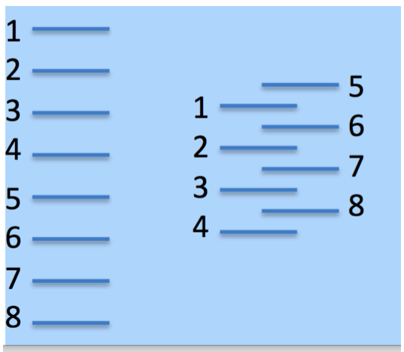


Figure 2: Initial deck (left), in-shuffle (right).

5. Determine what happens to the card at rank k
6. If $2n + 1$ is prime, which is the case for a deck of 52 cards, use the Fermat little theorem to show that the number of rounds of in-shuffles required for getting back to the initial position divides $p - 1$.

With some training, you are now ready to make the trick for your Christmas party!