UE Mathematics for Computer Science

First session exam December 8, 2022 (3 hours)

Important information.
Read this before anything else!

- ▷ Only one personal hand-written sheet (2 pages) is allowed.
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ Please write your answers to each part on **separate sheets of papers** (2 separate sheets corresponding to Part A and Part B of the exam).
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All answers should be well-argued to be considered correct.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

Indicative grades

	Part A				Part B			CWQP
Exercises	I.1	I.2	II.1	II.2	III.1	III.2	IV	
points	2.5	3	2,5	2	2	2	7	2

CWQP is for **C**larity of **W**riting and **Q**uality of **P**resentation.

Part A: Proofs, Recurrences and Structures

I. Variations on Pascal's triangle

Elements that have been developed during the lectures and could be useful for the next questions.

Pascal's Triangle

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Fermat's Little Theorem

Let a be an integer, and let p be any fixed prime number. Then:

$$a^p \equiv a \text{ modulo } p$$

Another way to state the theorem is :

For any integer a, and for any prime p,
the number $a^p - a$ is divisible by p

I.1. **Pascal's triangle and Fermat's Little Theorem**

- I.1.a. Draw the Pascal's triangle modulo 7
- I.1.b. Prove by induction on integer a that $a^p \equiv a \bmod p$.
  Hint: you may invoke here a restricted form of the Binomial Theorem,

I.2. **Some properties of Pascal's triangle**

- I.2.a. Prove that each row of the triangle is a palindrome.
- I.2.b. Prove that each diagonal sums up to the successive Fibonacci numbers.
- I.2.c. Prove that the third column corresponds to the triangular number Δ_n.

II. Tribonacci

Reading the classical Charles Darwin's *Origin of Species* book, there exists a similar relation to elephant growth of populations as the Fibonacci numbers do to rabbits. The dynamic of births lead to the following tri-linear progression $T(n)$.

Given the three numbers $T(0) = 0$ and $T(1) = T(2) = 1$

$$T(n) = T(n-3) + T(n-2) + T(n-1) \text{ for } n \geqslant 3$$

II.1. **Structural property**

- II.1.a. Compute the first rank of the progression up to $T(9)$
- II.1.b. Prove the following proposition:

$$T(n+1) = \frac{T(n+2) + T(n-2)}{2}$$

II.2. **Interpretation**

- II.2.a. Show that $T(n)$ is the number of different ways to write $n-2$ with no term greater than 3.
  For instance, for $n = 5$, $T(5) = 4$ and $n - 2 = 3$ can be written as: $1+1+1$, $1+2$, $2+1$ and 3
- II.2.b. (it is a hard question) Show that the n-th element of this progression corresponds to the number of ordered trees with $n+1$ edges with all leaves at level 3.

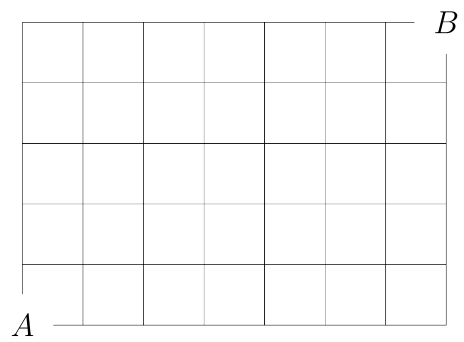
Part B : Combinatorics and Probabilities

III. Exercises

III.1. Prove the following identity:

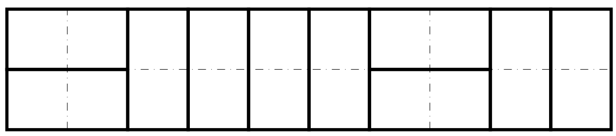
$$\sum_{k=0, k \text{ even}} \binom{n}{k} = \sum_{k=0, k \text{ odd}} \binom{n}{k} \text{ for all } n \geqslant 1$$

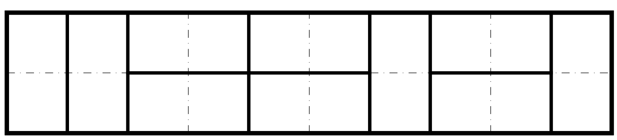
III.2. In a rectangle grid $m \times n$ compute the number of shortest paths from $A = (0, 0)$ to $B = (m, n)$.



IV. Problem on friezes

We consider tiles with size 2×1. A frieze with height 2 , consists in a succession of tiles put horizontally or vertically without holes. Here are two examples of friezes of length 10.





The aim of this problem is to build a random frieze of 2×1 tiles of length 20.

IV.1. Propose an algorithm that generates a random frieze.

Comment: detail and justify the approach, give a proof of the algorithm, evaluate its complexity, give examples, provide all the elements to convince on the quality of the proposed solution