UE Mathematics for Computer Science

First session exam December 7, 2022 (3 hours)

Important information.
Read this before anything else!

- ▷ Only one personal hand-written sheet (2 pages) is allowed.
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ Please write your answers to each part on **separate sheets of papers** (2 separate sheets corresponding to Part A and Part B of the exam).
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All answers should be well-argued to be considered correct.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

Indicative grades

	Part A				Part B				
Exercises	I	II	III	IV	V.1	V.2	V.3	V.4	V.5
points	3	6	2	2	3	3	2	3	2

Part A : Exercises

I. Divisibility

Preliminary: The successive *remainders* from repeated Euclidian divisions of an integer n by a number-base b are the successive digits of the base b numeral for n, from the lowest-order digit to the highest.

In other words, if the base-b numeral for n is the string

$$\delta_m \delta_{m-1} \cdots \delta_1 \delta_0$$

then the successive remainders are, in order, δ_0, δ_1, $\ldots$, δ_{m-1}, δ_m.

This is well-known for base $b = 2$ (since it is the basis of the binary system).

I.1. Prove the following assertion.

An integer n is divisible by an integer m if, and only if, m divides the sum of the digits in the base-$(m+1)$ numeral for n.

I.2. Deduce the most familiar instance of this result for $m = 9$:

An integer n is divisible by 9 if, and only if, the sum of the digits of n's base-10 numeral is divisible by 9.

II. A property of Fibonacci numbers

We want to prove here that $GCD(F(n), F(m)) = F(GCD(n, m))$.
As an example: $GCD(F(12), F(18)) = GCD(144, 2584) = 8 = F(6)$ (where $6 = GCD(12, 18)$).

Let us first recall the definition and fix the notations:

$$F(n+1) = F(n) + F(n-1) \text{ for all } n \geqslant 2 \text{ with } F(1) = F(2) = 1$$

Without loss of generality, consider that $n \geqslant m$.

Let us denote $g = GCD(n, m)$ and $G = GCD(F(n), F(m))$.

II.1. Check the property on small ranks:
$n = 8$ and $m = 6$
$n = 10$ and $m = 5$.

Let us first prove three technical lemmas.

II.2. Prove the following relation for any integers n and $k \geqslant 1$

$$F(n+k) \;=\; F(k) \cdot F(n+1) \;+\; F(k-1) \cdot F(n)$$

This relation assumes that we are able to define negative Fibonacci numbers. There is a ”natural” way of extending the definition to negative numbers.

II.3. Prove that for any integer $k \geqslant 1$, $F(n)$ divides $F(k \cdot n)$

II.4. Prove that for any integer $n \geqslant 1$, $F(n)$ and $F(n-1)$ are relatively prime:

$$\text{GCD}(F(n-1), F(n)) \ = \ 1$$

We are able now to prove the final result:

II.5. Using the previous results, prove $F(g) = G$

III. An exercise on graph

III.1. Prove that $K_{3,2}$ is not outer-planar.

IV. Proving a geometric result

IV.1. Recall the geometric construction of the expression:

$$\sum_{0 \leqslant k \leqslant n} \frac{1}{2^k}.$$

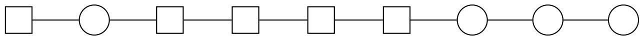
IV.2. Detail the proof when $n \to \infty$.

Part B : Problem

V. Problem on necklaces

V.1. A necklace with two types of jewels

Consider a linear necklace with size of n jewels composed with two types of jewels (sapphires and rubies).



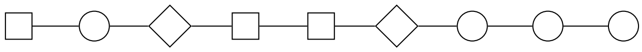
A necklace with 5 sapphires and 4 rubies.

V.1.a. Compute the number of different necklaces with n_S sapphires and n_R rubies ($n_S \geqslant 0$ and $n_R \geqslant 0$).

V.1.b. Compute the number of possibilities (n_1, n_2), for a given size n of the necklace. Compute this number under the supplementary constraint that a necklace with 2 types of jewels contains at least one jewel of each type.

V.1.c. Compute the number of necklaces with size n with no two consecutive sapphires.

V.2. A necklace with three types of jewels



A necklace with 3 sapphires, 4 rubies and 2 diamonds.

V.2.a. Prove that the number of different necklaces with n_1 sapphires and n_2 rubies and n_3 diamonds is:

$$\frac{(n_1 + n_2 + n_3)!}{n_1! n_2! n_3!}$$

V.2.b. A necklace with 3 types of jewels contains at least one jewel of each type. Compute the number of possibilities for (n_1, n_2, n_3), for a given size n of the necklace.

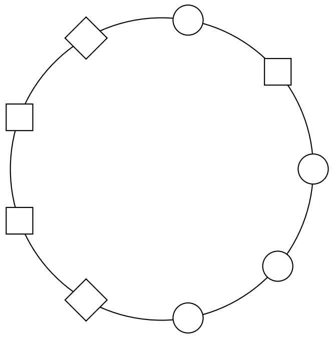
V.2.c. Propose an algorithm that generates uniformly a random linear necklace of size n and 3 types of jewels.

V.3. A necklace with k types of jewels

V.3.a. Compute the number of different necklaces with n_i jewels of type i.

V.3.b. An anagram of a word is a shuffle of its letters, `'bruy'` is an anagram of `'ruby'`. What is the link between the necklace problem and the anagrams ?

V.4. A circular necklace with n jewels and k types of jewels (research question)

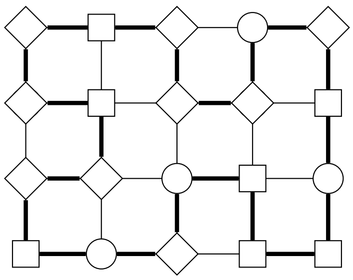


A necklace with 3 sapphires, 4 rubies and 2 diamonds.

We want to compute the number of such necklaces. Explain why this problem could be difficult. Are there some particular cases that are easy to solve ? How the size of the necklace give an information on the number of necklaces invariant by some rotation ?

V.5. **A grid of jewels**

Consider a 2-dimensional mesh of jewels (p, q) we want to extract an embedded circular necklace.



A grid of jewels $p = 5, q = 4$ and a possible embedded necklace (among many).

V.5.a. Show how to do when one (at least) of the p, q is even.

V.5.b. Show that it is impossible if p and q are both odd.