UE Mathematics for Computer Science

First session exam December 14, 2020 (3 hours)

Important information.
Read this before anything else!

- ▷ Only one personal hand-written sheet (2 pages) is allowed.
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ Please write your answers to each part on **separate sheets of papers** (2 separate sheets corresponding to Part A and Part B of the exam).
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All answers should be well-argued to be considered correct.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

Indicative grades

	Part A		Part B	
Exercises	I	II	III	IV
points	6	6	11	1

Part A : Count and Simulate

I. Derangements

A derangement on k distinct objects is a permutation of these k objects such that each object is not on its initial place. As an example, if we have the $k = 5$ objects $(x_1, x_2, x_3, x_4, x_5)$ a derangement is $(x_3, x_5, x_2, x_1, x_4)$. But $(x_2, x_1, x_3, x_5, x_4)$ is not a derangement because x_3 is in its initial position.

Denote by D_k the total number of derangements of k objects. It has been shown in a lecture that the D_k satisfy a recurrence relation :

$$D_k = (k-1)(D_{k-1} + D_{k-2}) \text{ with } D_0 = 1 \text{ and } D_1 = 0$$

It has been deduced that

$$D_k = k! \sum_{i=0}^{k} \frac{(-1)^i}{i!}.$$

I.1. Propose an algorithm that generates uniformly a random derangement of size k.

I.2. Evaluate the complexity of this algorithm

Consider the following algorithm

Generator(T**,**N**)**

Data: T an array of objects $T = [x_1, \cdots, x_k]$, k the size of the array
Result: A random permutation of objects in T after N iterations

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for n = 1 to N
    i ← RandomInt (1, k)
    j ← RandomInt (1, k)
    Exchange (T, i, j)
return T
```

I.3. Compute the probability π_n that object x_1 is in the first position after n iterations. What is the limit of π_n as n goes to infinity ?

Hint : Use the sequence of random variables Y_n defined by

$$Y_n = \begin{cases} 1 & \text{if } T[1] = x_1 \text{ at step } n \\ 0 & \text{elsewhere.} \end{cases}$$

then $\pi_n = \mathbb{P}(Y_n = 1 | Y_0 = 1)$.

II. Divisors

The aim of this exercise is to estimate the expected number of divisors of number k taken randomly and uniformly in $\{1, 2, \cdots, n\}$.

Define by $\tau(k)$ the numbers of divisors of k. As an example $\tau(12) = 6$, because one can divide 12 by 1, 2, 3, 4, 6 and 12, (6 divisors)

II.1. Plot the function $\tau(k)$ for $k = 1, \cdots, 20$. What could be observed on this curve ? What is the limit of $\tau(k)$ when k grows to infinity ?

II.2. If k is a power of m compute $\tau(k)$. Deduce that there are subsequences of $\{\tau(k)\}_{k\in\mathbb{N}}$ growing to infinity.

Consider the unique decomposition of $k = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$ where $p_1, p_2, \cdots, p_l$ are distinct prime numbers.

II.3. From this decomposition

II.3.a. Compute $\tau(k)$

II.3.b. Prove that k is a perfect square if and only if $\tau(k)$ is odd.

II.3.c. Deduce that the τ function is multiplicative, that is

for all m, n prime together integers $\tau(mn) = \tau(m)\tau(n)$

II.4. Prove the following identity :

$$\sum_{k=1}^{n} \tau(k) = \sum_{l=1}^{n} \left[\frac{n}{l}\right] \text{ where } [x] \text{ is the integer part of } x, \quad [x] \leqslant x < [x] + 1$$

Hint : consider the $n \times n$ matrix $A = ((a_{i,j}))$ where coefficient $a_{i,j} = 1$ if i divides j and 0 if not.

II.5. Deduce that

$$\frac{1}{n} \sum_{k=1}^{n} \tau(k) \simeq \log n \text{ for } n \text{ large,}$$

and answer the initial question.

Part B : Structural Properties

III. Some properties of the Pascal's triangle

III..1 Preliminaries

We start by recalling pictorially the construction of the Pascal's triangle.

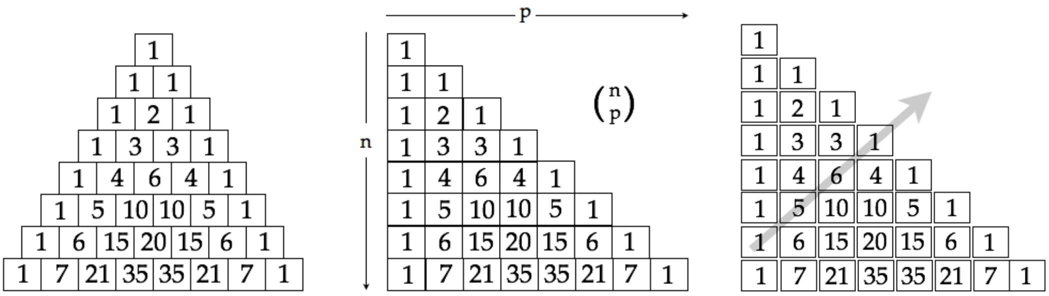

Figure 1: First rows of the Pascal's triangle, the representation by columns of the binomial coefficients, and a diagonal in the triangle.

III.1. Show the following basic properties.

III.1.a. Prove that each row of the triangle is a palindrome.

III.1.b. Show mathematically that each diagonal sums up to the successive Fibonacci numbers.

III..2 The Magical Sequence

The magical sequence studied during the homework is known as the Stern's sequence. It is recalled as follows:

$s_0 = 0$ and $s_1 = 1$ initial values;

$$\begin{cases} s_{2k} = s_k & \text{even case;} \\ s_{2k+1} = s_{k+1} + s_k & \text{odd case.} \end{cases}$$

This sequence has many links with the Fibonacci sequence. In particular, we are interested in the links between s_k and the Pascal's triangle.

The s_k are obtained using a property similar to Fibonacci, by summing the number of odd numbers in the successive diagonals (figure 2).

III.2. Computation of the s_k

III.2.a. Prove first this property for each row 2^q.
Then, prove the property for any row.

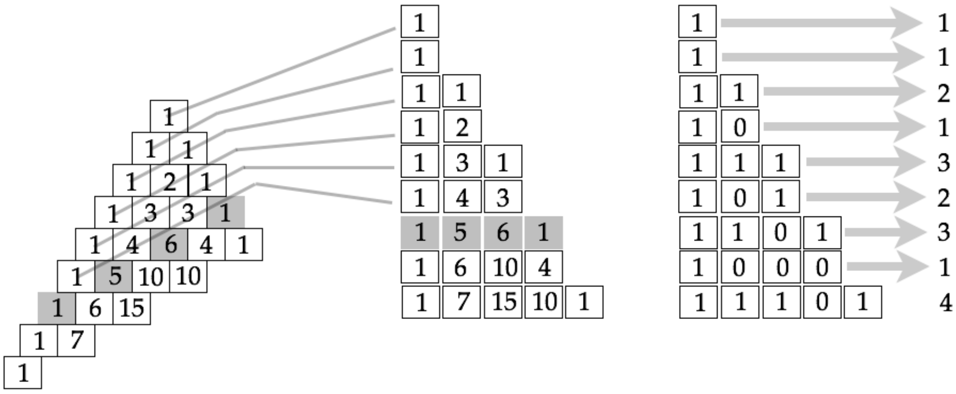

Figure 2: Number of odds on the diagonals.

III.2.b. Deduce a closed form of the s_k.
Hint: express the indices of the shaded elements in the triangle as an affine function in n and p.

III..3 Combinatorial proof

III.3. Show that $s(n)$ can be interpreted as the number of different representations of the integer $n-1$ in the *extended binary notation*, that is obtained by sums of powers of 2 taken 0, 1 or 2 times[1]

Hint: we can decompose the analysis in even and odd cases.

III..4 Computing a summation of binomial coefficients

Let $Q_n = \Sigma_{k \leqslant 2^n} (-1)^k \times \binom{2^n - k}{k}$. We are looking for an expression of Q_n by induction.

III.4. Computation of $Q_{1,000,000}$

III.4.a. Compute the first ranks (for $n = 1, 2, 3$ and 4).

III.4.b. As binomial coefficients are involved, a good idea is to use Pascal's triangle.
However, $2^n - k$ is 2^k rows apart k, thus, we can not easily use the locality character of the triangle.
Instead, we change by computing $R_m = \Sigma_{k \leqslant m} (-1)^k \times \binom{m-k}{k}$
by using the relation $\binom{m-k}{k} = \binom{m-(k+1)}{k} + \binom{m-(k+1)}{k-1}$

III.4.c. Compute the first ranks and show $R_m = R_{m-6}$

III.4.d. Deduce the expression of $Q_{1,000,000}$

IV. A Path and Graph Question

IV.1. If every vertex of graph G has degree $\geqslant d$, then G contains an elementary path of length d.

[1] for instance, $8 = 9 - 1$ can be decomposed in four different ways by: $8 = 1000_2$, $4 + 4 = 200_2$, $4 + 2 + 2 = 120_2$, $4 + 2 + 1 + 1 = 112_2$