UE Mathematics for Computer Science

First session exam December 13, 2019 (3 hours)

Important information.
Read this before anything else!

- ▷ Only one personal hand-written sheet (2 pages) is allowed.
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ Please write your answers to each part on separate sheets of papers (2 separate sheets).
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged.**

Indicative grades

	Part A				Part B	
Exercises	I	II.1	II.2	II.3	III	IV
points	5	3	2	3	5	7

Part A : Proofs, Recurrences, and Graphs

I. Planar graphs

Preliminary: outerplanar graphs

We recall below the definition of outerplanar graphs:

A graph G is outerplanar if it can be drawn by placing its vertices along a circle in such a way that its edges can be drawn without any crossings.

I.1. Show the following negative result:

$K_{3,2}$ is not outerplanar.

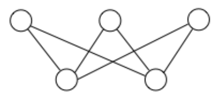

Figure 1: $K_{3,2}$.

A condition for planarity in graphs

Let consider a graph G of order n (it has n vertices) with e edges, and f faces.

A *face* in a drawing of G is a polygon whose sides are edges of G , whose points are vertices of G, and whose interiors are empty (it has no crossing edges).

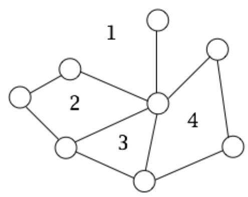

Figure 2: Example of a (planar) graph with 8 vertices, 10 edges and 4 faces.

The aim of the proposed exercice is to prove the classical Euler formula for planar graphs. Given an drawing of G, the following expression is true:

$$n - e + f = 2 \tag{1}$$

We will prove it by two different methods.

I.2. Solving by induction. A first approach validates (1) by growing a planar graph G inductively, edge by edge starting from a base case.

Prove the Euler formula (1) and detail the induction step (that is, adding a new edge) by a case by case analysis.

I.3. The other approach is to validate the expression via *deconstruction*.

The deconstruction of G is a step by step process of removing one edge. Show that this process preserves an *invariant* in expression (1) and conclude.

II. Mixed exercises

A fun result dealing with divisibility

This exercice was a favorite question by the famous mathematician Paul Erdos, he often used it to test the ability of young students in mathematics...

The problem is described as follows: Let consider the $2n$ first integers.

II.1. Take any $n+1$ integers in this set and prove that there exists a pair (p, q) such that p divides q.

Hint : Write the $2n$ numbers by decomposing the sequence into multiples of powers of 2. When there are multiple ways, we take the one with the largest power of 2 in order to make the decomposition unique. Example for $n = 7$:

1, 3, 5, 7, 9, 11, 13
2, 10, 14
4, 12
8

Thus, according to such a decomposition, all the integers of the sequence are written as: $2^k \times m$ where m is odd (and $k \geqslant 0$).

Use the pigeon hole principle to exhibit p and q.

Geometrical proof

II.2. Compute the sum of $(\frac{1}{4})^k$ using a graphical argument ($k \geqslant 0$).

Hint : A rapid analysis of the small values of k leads to the guess $\sum(\frac{1}{4})^k = \frac{1}{3}$. Actually, the expression of the sum of a geometrical sequence may also be applied here.

For the graphical construction: Assuming the total area of the isosceles triangle of Fig. 3 is 1, the area of the grey internal triangle (left) is $\frac{1}{4}$...

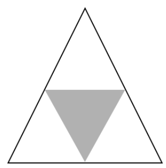

Figure 3: Basis of a possible graphical construction.

On adjacency matrix

II.3. Consider a graph and its adjacency matrix.

II.3.a. Give the adjacency matrix of the graph of Fig. 2.

II.3.b. Compute A^2.

To what property corresponds the graph of this adjacency matrix?

II.3.c. Deduce a way to prove the connectivity of a graph using the powers of the adjacency matrix.

II.3.d. Let now consider another type of operations instead of the classical arithmetic operations + and $\times$: boolean OR and boolean AND.

Give an interpretation of the adjacency matrix A^2 according to these operations.

Part B : Counting and Coin Tossing

III. Cycles in permutations

Consider $\mathcal{S}_n$ the set of all permutations on n objects. It has been shown in the homework that a permutation could be uniquely decomposed in set of disjoint cycles.

III.1. Recall the number of permutations in $\mathcal{S}_n$ with only one cycle and prove it.

III.2. Compute the number of permutations having exactly two cycles.

III.3. Write an algorithm that generates uniformly a random permutation with exactly 2 cycles. Prove the algorithm and evaluate its complexity.

III.4. For large n compute the probability that a random permutation in $\mathcal{S}_n$ has exactly two cycles.

IV. Coin tossing

Consider a simple coin tossing game : you have an initial fortune of n and you play against the bank. You flip a fair coin if the result is *Head* you win 1 and if the result is *tail* you loose 1. You win the game if you reach a fortune target N and loose the game if your fortune reaches 0.

IV.1. Propose a stochastic model for this problem. All structural and statistical assumptions should be detailed carefully.

The objective is to compute the probability to win the game when your initial fortune is n and your target is a fortune of N, $0 \leqslant n \leqslant N$

IV.2. Solve the problem for $N = 5$ and $n = 3$.

For a given N denote by q_n^N the probabilities to win the game when the initial fortune is n.

IV.3. Prove that the q_n^N satisfy the linear system of equations :

$$\left\{ \begin{array}{lcl} q_0^N & = & 0, \\ \vdots & & \\ q_n^N & = & \frac{1}{2} q_{n-1}^N + \frac{1}{2} q_{n+1}^N \quad \text{for } 0 < n < N, \\ \vdots & & \\ q_N^N & = & 1. \end{array} \right.$$

IV.4. Compute the solution of this system and interpret it.

Denote by τ_n^N the expected length of the game for an initial fortune of n.

IV.5. Establish a linear system satisfied by the τ_n^N. (The solution is **not** asked)

IV.6. What is modified in the answers of all the previous questions if the coin is unfair and p is the probability to get *Head*?