Maths for Computer Science
Correction of the Quizz – Part 1

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Geometric series

let n be an integer, $\Sigma_{k=0,n} 2^k = ?$

This is a particular case ($a = 2$) of the **geometric progression**.
$S_a(n) = \Sigma_{k=0,n} a^k = \frac{a^{n+1}-1}{a-1}$ for $a \neq 1$

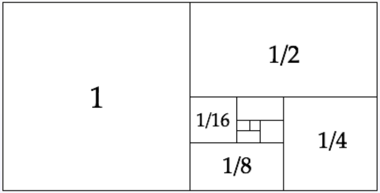
Let us expand the summation:

- $S_a(n) = 1 + a + a^2 + \cdots + a^n$
- $= 1 + a[1 + a + a^2 + \cdots + a^{n-1}] + a^{n+1} - a^{n+1}$
- $= 1 + a \cdot S_a(n) - a^{n+1}$
- Thus, $(1 - a)S_a(n) = 1 - a^{n+1}$

Remark that most existing proofs directly suggest to multiply $S(n)$ by $1 - a$

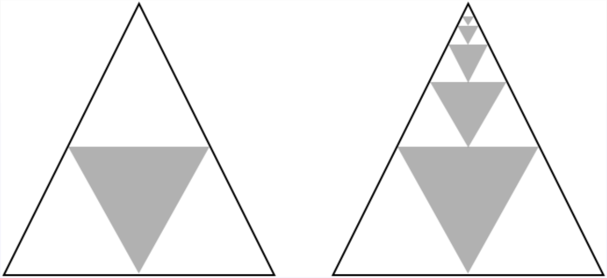
Particular geometric series

$a = \frac{1}{2}$



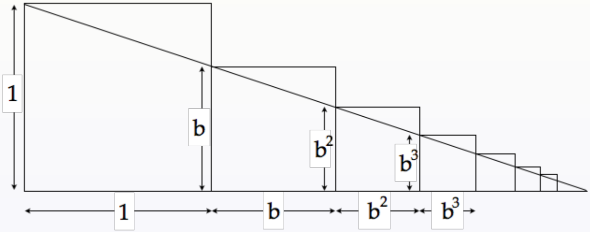
Particular geometric series

$a = \frac{1}{4}$



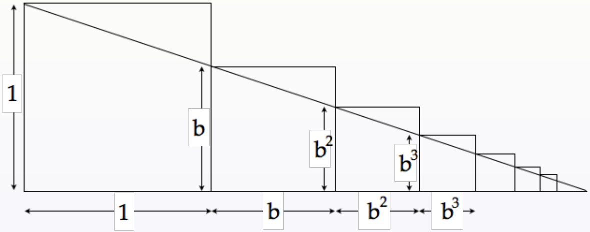
- Assuming the base triangle area is 1, the solution is the grey area.
  It is one third at each layer: $S_{1/4} = \frac{1}{3} + 1 = \frac{4}{3}$

Any geometric series with $b < 1$



[1] notice here the transversality of the topics in Maths

Any geometric series with $b < 1$



- The value of the summation is given by the Thales' theorem (triangle similarity)[1]:
  $\frac{S_b}{1} = \frac{1}{1-b}$

[1] notice here the transversality of the topics in Maths

Proving an expression with a summation

Prove
$\Sigma_{k=1,n}[k^2(k+1) - k(k-1)^2] = n^2(n+1)$ for all non-negative integer n

The idea here is to **figure out how to write a simple proof**.

To get insight, let start by small values of n.

- $n = 1$, $1^2(1+1) - 1(1-1)^2 = 1^2(1+1)$
- $n = 2$,
  $1^2(1+1) - 1(1-1)^2 + 2^2(2+1) - 2(2-1)^2 = 2^2(2+1)$

In both cases, we see that the sum reduces to a single term...

Proving this result

The summation can be written as follows:
$n^2(n+1) + \Sigma_{k=1,n-1}(k^2(k+1) - \Sigma_{k=1,n} k(k-1)^2)$

Now, let us remark that the last term can be simplified since the first term of this summation is nul for $k = 1$:
$\Sigma_{k=2,n} k(k-1)^2$

Now, shift the indices in this sum (change k to $k' = k + 1$):
$\Sigma_{k'=1,n-1}(k'+1)k'^2$.

This conclues the proof since both summations are the same with an opposite sign[2].

[2] notice here that a proof by expending the expression is also possible

Identities

$a^n - b^n = ?$
$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + b^{n-1})$.

The proof technique works exactly as before by cancelling pairs of equal terms!
The expression of question 1 above is obtained for $a = 2$ and $b = 1$.

$(a+b)^n = ?$
The second one is the classical Newton binomial expression.

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Harmonic series

What are the values of $\Sigma_{k\geq 0}\frac{1}{2^k}$ and $\Sigma_{k>0}\frac{1}{k}$?

$\Sigma_{k>0}f(k) = lim_{n\rightarrow\infty}\Sigma_{k=1,n}f(k)$
Obtaining a finite value for an infinite sum was a paradox for a long time until the infinitesimal calculus of Leibniz/Newton on the XVIIth century.

- The *limit* of the first sum is 2.
  This is obtained by using the sum of a geometric progression for $a = \frac{1}{2}$.
- The second sum is unbounded (it goes to $+\infty$).
  The result is obtained by bounding the summation:
  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... > 1 + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} + ...$
  and the infinite sum of positive constant numbers (here $\frac{1}{2}$) is infinite.

An extra (related) question

Compute extended geometric series and their sums
$S_a^{(c)}(n) = \sum_{i=1}^{n} \; i^c a^i$

We now build on our ability to evaluate geometric summations of where c is an arbitrary fixed positive integer, and a is an arbitrary fixed real number.

We restrict attention to summations $S_a^{(c)}(n)$ that satisfy the joint inequalities $c \neq 0$ and $a \neq 1$.

- We have already adequately studied the case $c = 0$, which characterizes "ordinary" geometric summations.
- The degenerate case $a = 1$ removes the "geometric growth" of the sequence underlying the summation.

Summation method

- The method is *inductive in parameter c*, for each fixed value of c, the method is *inductive in the argument n*.
  we restrict here to the case $c = 1$.

The summation $S_a^{(1)}(n) = \sum_{i=1}^{n} ia^i$

Proposition.

For all bases $a > 1$,

$$S_a^{(1)}(n) \quad = \quad \sum_{i=1}^{n} ia^i \quad = \quad \frac{(a-1)n-1}{(a-1)^2} \cdot a^{n+1} + \frac{a}{(a-1)^2} \qquad (1)$$

Proof

We begin to develop our strategy by writing the natural expression for $S_a^{(1)}(n) = a + 2a^2 + 3a^3 + \cdots + na^n$ in two different ways.

First, we isolate the summation's last term:

$$S_a^{(1)}(n+1) = S_a^{(1)}(n) + (n+1)a^{n+1} \tag{2}$$

Then we isolate the left-hand side of expression:

$$\begin{aligned} S_a^{(1)}(n+1) &= a + \sum_{i=2}^{n+1} ia^i \\ &= a + \sum_{i=1}^{n} (i+1)a^{i+1} \\ &= a + a \cdot \sum_{i=1}^{n} (i+1)a^i \end{aligned}$$

Proof

$$
\begin{aligned}
&= a + a \cdot \left(\sum_{i=1}^{n} ia^i + \sum_{i=1}^{n} a^i \right) \\
&= a \cdot \left(S_a^{(1)}(n) + S_a^{(0)}(n) \right) + a \\
&= a \cdot \left(S_a^{(1)}(n) + \frac{a^{n+1} - 1}{a - 1} - 1 \right) + a \\
&= a \cdot S_a^{(1)}(n) + a \cdot \frac{a^{n+1} - 1}{a - 1} \qquad (3)
\end{aligned}
$$

Combining both previous expressions of $S_a^{(1)}(n+1)$, we finally find that

$$\begin{aligned}(a-1)\cdot S_a^{(1)}(n) &= (n+1)\cdot a^{n+1} \;-\; a\cdot\frac{a^{n+1}-1}{a-1}\\ &= \left(n-\frac{1}{a-1}\right)\cdot a^{n+1} \;+\; \frac{a}{a-1} \qquad (4)\end{aligned}$$

We now use standard algebraic manipulations to derive the expression

Solving the case $a = 2$ using subsum rearrangement.
We evaluate the sum $S_2^{(1)}(n) \ = \ \sum_{i=1}^{n} \ i2^i$
in an especially interesting way, by rearranging the sub-summations of the target summation.

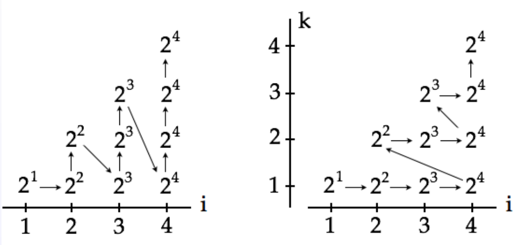
Underlying our evaluation of $S_2^{(1)}(n)$ is the fact that we can rewrite the summation as a *double* summation:

$$S_2^{(1)}(n) \ = \ \sum_{i=1}^{n} \sum_{k=1}^{i} 2^i \qquad (5)$$

By suitably applying the laws of arithmetic specifically, the distributive, associative, and commutative laws, we can perform the required double summation in a different order than that specified previously.

We can exchange the indices of summation in a manner that enables us to compute $S_2^{(1)}(n)$ in the order implied by the following expression:

$$S_2^{(1)}(n) \;=\; \sum_{k=1}^{n} \sum_{i=k}^{n} 2^i$$



Expanding these instances, we find finally that

$$\begin{aligned} S_2^{(1)}(n) &= \sum_{k=1}^{n} \left(2^{n+1} - 1 - \sum_{i=0}^{k-1} 2^i\right) \\ &= \sum_{k=1}^{n} \left(2^{n+1} - 2^k\right) \\ &= n \cdot 2^{n+1} - (2^{n+1} - 1) + 1 \\ &= (n-1) \cdot 2^{n+1} + 2 \end{aligned}$$

We remark that the process of obtaining the original summation can also be seen in the figure, by scanning the elements of the summation along diagonals.
Each of the n diagonals contains exactly the difference between the complete geometric summation and the partial summation that is truncated at the kth term.