Master of Science in Informatics at Grenoble

UE Mathematics for Computer Science

Mid-term exam October 22, 2024 (1 hour)

Important information.
Read this before anything else!

- ▷ No document allowed
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All answers should be well-argued to be considered correct.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points allotted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

Indicative grades

	Exercises			Problem	CWQP
Exercises	I	II	III	IV	
points	2	3	5	10	2

CWQP is for **C**larity of **W**riting and **Q**uality of **P**resentation.

I. Exercise : a graphical proof

I.1. Compute by a graphical proof the sum of

$$\sum_{k=0}^{n} \left(\frac{1}{4}\right)^k .$$

II. Exercise: Stern's sequence

Let consider the Stern sequence s whose definition is recalled below:

$s(0) = 0$ and $s(1) = 1$

$s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$ for all $n \geqslant 2$

II.1. Show for all odd index i,

$$s(i) = s(i-1) + s(i+1).$$

II.2. Deduce for any row of the array (as shown in lectures), the maximum is a Fibonacci number.

Hint: Characterize its index

III. Exercise: Tribonacci

Reading the classical Charles Darwin's *Origin of Species* book, there exists a similar relation to elephant growth of populations as the Fibonacci numbers do to rabbits.

The dynamic of births lead to the following tri-linear progression:

Given the three numbers $F(1) = 0$ and $F(2) = F(3) = 1$

$F(n) = F(n-3) + F(n-2) + F(n-1)$ for $n \geqslant 4$

III.1. Compute the first ranks of the progression up to $F(9)$

III.2. Prove the following proposition:

$$F(n+1) = \frac{F(n+2) + F(n-2)}{2}$$

Interpretation

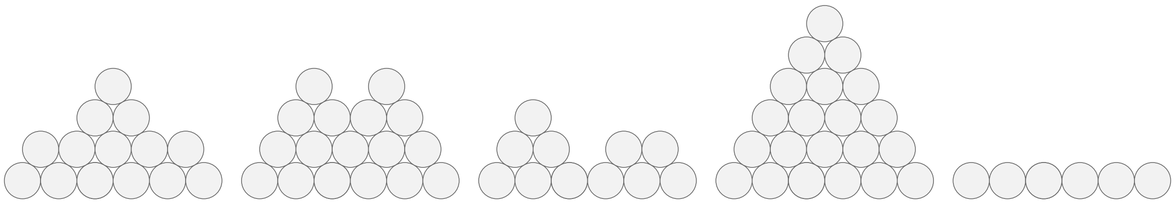
III.3. Show that $F(n)$ is the number of different ways to write $n-2$ with no term greater than 3.

For instance, for $n = 5$, $F(5) = 4$ and $n - 2 = 3$ can be written as: $1+1+1$, $1+2$, $2+1$ and 3

III.4. (Bonus) *a hard question* Show that the n-th element of this progression corresponds to the number of ordered trees with $n+1$ edges with all leaves at level 3.

IV. Pyramids of coins

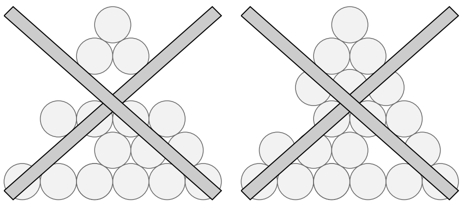
We construct heaps of coins by first putting a base and then placing coins by level such that a coins is always put on top of two adjacent coins.



(a) An example (b) Another example (c) Another example (d) Another example (e) Another example

Figure 1: Some examples of coins heaps with a base of size 6

In figure 1 the size of base is 6, the two last cases are extremal cases with a maximum number of coins (1d) and a minimal number (1e) with a base of 6.



(a) (b)

Figure 2: Two situations that are not heaps of coins with a base of size 6

IV.1. Compute the number of possible heaps of coins with a base of 7.

IV.2. Propose an algorithm that enumerates all the possible heaps of coins with a given base.

Comment: For this research activity, explain your approach, give justifications to the methods and algorithms, evaluate the complexities, give examples, provide all the elements to convince on the quality of the proposed solution