# Lecture 5 – Maths for Computer Science
# Basic on graphs
# Part. 2: Structured Graphs

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november 2024

## Objectives

Review some structured graphs and study their properties.

- Complete graphs
- Cycles
- Meshes and torus
- Hypercubes
- Trees

## Complete graphs (or cliques)

Definition.
Each vertex of $K_n$ is connected to all the other vertices.

- Connected ($D = 1$)
- Regular graph ($\delta = n - 1$)
- Number of edges $\Sigma_{1 \leq k \leq n-1}\ k = \frac{(n)(n-1)}{2}$

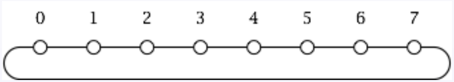
## Rings or Cycles

### Definition.

Each vertex of $C_n$ has exactly one predecessor and one successor.

### Coding of edges

$$\{\{i,\ i+1 \bmod n\} \mid i \in \{0,\ 1,\ \ldots,\ n-1\}\}.$$



- Connected ($D = \lfloor \frac{n}{2} \rfloor$)
- Regular graph ($\delta = 2$)
- Number of edges $n$

## An interesting observation

Proposition.

If every vertex of a graph $G$ has degree $\geq 2$, then $G$ contains a cycle.

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**Proposition.**
If every vertex of a graph $G$ has degree $\geq 2$, then $G$ contains a cycle.

**Proof**
Let us assume by contradiction that we have a cycle-free graph $G$ all of whose vertices have degree $\geq 2$.

Let us view graph $G$ as a park where every vertex of $G$ is a statue, and every edge is a path between two statues.
The fact that every vertex of $G$ has degree $\geq 2$ means that if we take a stroll through $G$, then every time we leave a vertex $v \in V$, we can use a *different* edge/path than we used when we came to $v$.

## Meshes and Torus

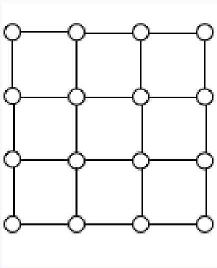
Definition.
Cartesian product of paths/cycles.

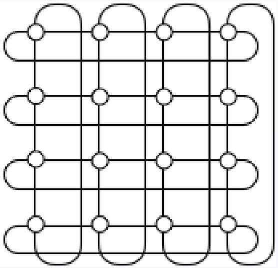
Coding of meshes (vertices and edges).

$$\{1,\ 2, \ldots,\ m\} \times \{1,\ 2, \ldots,\ n\}$$
$$\left\{\langle i,\ j\rangle \ \mid\ \left[i \in \{1,\ 2, \ldots,\ m\}\right],\ \left[j \in \{1,\ 2, \ldots,\ n\}\right]\right\}$$

Torus is obtained by adding the wraparound links...

## Example for $n = 4$





## Properties of the square torus with $n$ vertices

$\sqrt{n}$ by $\sqrt{n}$

- Connected (diameter $D = \Theta(\sqrt{n}) = 2 \cdot \lfloor \frac{\sqrt{n}}{2} \rfloor$)
- Regular graph (degree $\delta = 4 = \Theta(1)$)
- Number of edges $2n = \Theta(n)$

## Hypercubes

Motivation:
build a graph with a trade-off beetwen the degree and the diameter.

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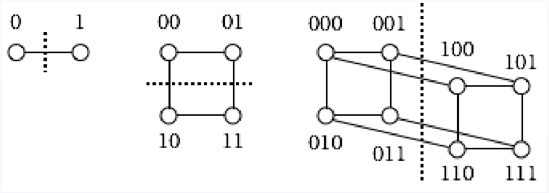
Recursive Definition.

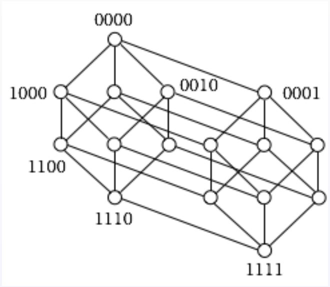
- The order-0 boolean hypercube, $H_0$, has a single vertex, and no edges.
- The order-$(k+1)$ boolean hypercube, $H_{k+1}$, is obtained by taking two copies of $H_k$ ($H_k^{(1)}$ and $H_k^{(2)}$), and creating an edge that connects each vertex of $H_k^{(1)}$ with the corresponding vertex of $H_k^{(2)}$.

## Construction



## The next dimension

Representation of $H_4$



## Coding

### A natural binary coding

The coding from the vertices is naturally in the binary system.

The coding of two adjacent vertices is obtained by flipping only one bit.

## Characteristics of Hypercubes

The number of vertices is a power of 2: $n = 2^k$ $(k = log_2(n))$

- Diameter $D_n = k$
- Degree $\delta_n = k$
- Number of edges?

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- Diameter $D_n = k$
- Degree $\delta_n = k$
- Number of edges?

$H_{k+1}$ is obtained by two copies of $H_k$ plus $2^k$ edges for linking each relative vertex, thus:
$N_{k+1} = 2 \times N_k + 2^k$ starting at $N_0 = 0$

$N_k = k \times 2^{k-1}$

## Graph isomorphism

The difficulty here is that there are many ways to draw a graph...

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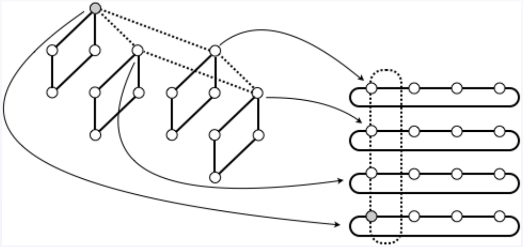
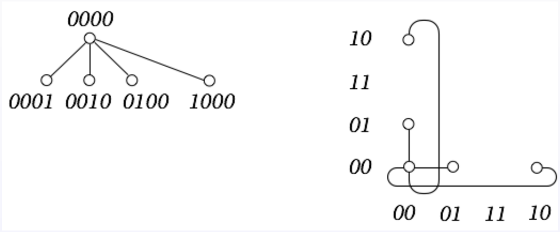
The proof is by an adequate coding of the vertices/edges.

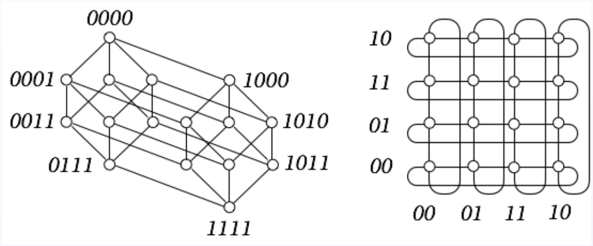


## Coding schemes

The following figure (left) depicts this coding of a vertex and its neighbors.



## The (almost) full picture



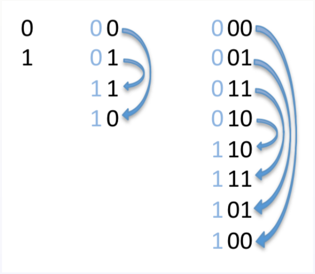
## Gray Codes

### Cultural aside

Let us present the most popular code, namely, the **Reflected Gray code**

The 1-bit Gray code is simply 0 and 1.
The next one (for 2-bits) is obtained by mirroring the 1-bit code and prefix it by 0 and 1.
The next ones are obtained similarly.

## Gray Code

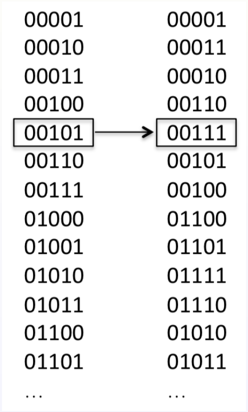


How to obtain the Gray code from the binary code?

Gray code can easily be determined from the classical binary representation as follows
$(x_{n-1}x_{n-2}...x_1x_0)_2$
shift right:
$(0x_{n-1}x_{n-2}...x_1)_2$

Take the exclusive OR (bit-to-bit) between the binary code and its shifted number:
$(x_{n-1}(x_{n-2} \oplus x_{n-1})...(x_0 \oplus x_1))_G$

For instance the binary code of $5 = (00101)_2$ is
$(0 \oplus 0)(0 \oplus 0)(0 \oplus 1)(1 \oplus 0)(0 \oplus 1) = (00111)_G$.



| | |
|---|---|
| 00001 | 00001 |
| 00010 | 00011 |
| 00011 | 00010 |
| 00100 | 00110 |
| $\boxed{00101}$ $\longrightarrow$ | $\boxed{00111}$ |
| 00110 | 00101 |
| 00111 | 00100 |
| 01000 | 01100 |
| 01001 | 01101 |
| 01010 | 01111 |
| 01011 | 01110 |
| 01100 | 01010 |
| 01101 | 01011 |
| … | … |

## Matching

Definition

A matching is a set of edges that have no vertices in common.

It is *perfect* if its vertices are all belonging to an edge[1].

Proposition.

The number of perfect matchings in a graph of order $n = 2k$ grows exponentially with $k$.

[1] thus, the number of vertices is even and the cardinality of the matching is exactly half of this number

## Example

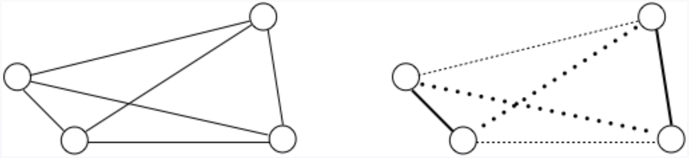


Figure: The 3 possible perfect matchings of $K_4$.

## Proof

by recurrence on $k$,
let denote the number of perfect matchings by $N_k$.

**Base case:** For $k = 1$, there is only one perfect matching $N_1 = 1$ and for $k = 2$, there are 3 different perfect matchings $N_2 = 3$.
**Induction step:** For $k$, there are $2k - 1$ possibilities for a vertex to choose an edge, $N_k = (2k - 1).N_{k-1}$
Thus, $N_k$ is the product of the $k$ first odd numbers.

However, determining a perfect matching of minimal weight in a weighted graph can be obtained in polynomial time (using the Hungarian assignment algorithm).

## Another interesting class of graphs.

### Bipartite graphs.

A graph $G$ is bipartite if its vertices can be partitioned into (by definition of “partition”, disjoint) sets $X$ and $Y$ in such a way that every edge of $G$ has one endpoint in $X$ and the other in $Y$.

An interesting question is to link bipartite graphs and matchings.

## Trees

### Definition

Trees are identified mathematically as graphs that contain no cycles or, equivalently, as graphs in which each pair of vertices is connected by a unique path.

A tree is thus the embodiment of “pure” connectivity, which provides the minimal interconnection structure (in number of edges) that provides paths that connect every pair of vertices.

### Proposition

Any tree of order $n$ has $n-1$ edges.

## Example

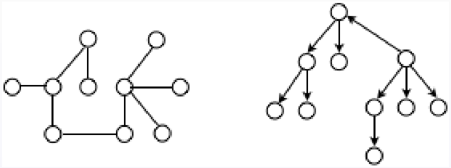


Figure: Undirected and directed trees.

## Preliminary results

### Lemma

1. Let $G$ be a connected graph with $n \geq 2$ vertices.
   Every vertex of $G$ has degree at least 1.
2. Any connected tree of order $n$ ($n \geq 1$) has at least one vertex of degree 1 (called a leaf).

## Preliminary results

### Lemma

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### Rapid Proofs

The main argument is on the analysis of graphs with minimum degrees 0 for part 1 and more than 2 for part 2.

## Proof 1

**Principle**

By induction on the order of the graph $n$.

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By induction on the order of the graph $n$.

**Base case** for $n = 2$
**Induction step.** Use the previous Lemma.

## Proof 2

**Inductive hypothesis**. Assume that the indicated tally is correct for all trees having no more than $k$ vertices.

**Inductive extension**. Consider a tree $T$ with $k+1$ vertices.

- By the Lemma, $T$ must contain at least one vertex $v$ of degree 1.
- If we remove $v$ and its (single) incident edge, we now have a tree $T'$ on $k$ vertices.
- By induction, $T'$ has $k-1$ edges. When we reattach vertex $v$ to $T'$, we restore $T$ to its original state.
- Because this restoration adds one vertex and one edge to $T'$, $T$ has $k+1$ vertices and $k$ edges.

## Spanning Trees

Let consider a weighted graph $G$.

### Motivation

a way of succinctly “summarizing” the connectivity structure inherent in undirected graphs.

### Definition

Take the same set of vertices and extract a set of edges that spans the vertices.

- Determining a minimal Spanning Tree is a polynomial-time problem.
- There exist two possible constructions, following two different philosophies (following each one feature of the problem, namely (i) to obtain a tree and (ii) to guarantee the minimum weight).