Lecture 5 – Maths for Computer Science
Basics on Graphs (part 1)

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Lecture notes MoSIG1

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Objective and Plan

The purpose of this lecture is to present discrete structures and pathways to operate on such structures.

Definition.
Graphs represent sets of mathematical *objects* in relation.

There are many examples in the *real life*.

- Part 1 deals with basic definitions and concepts.
- Part 2 provides the analysis of properties on structured graphs

What is a graph?

Recall: A graph is a finite set of vertices linked according to a given *relation*.

Formally

A graph G is defined as the pair (V, E).

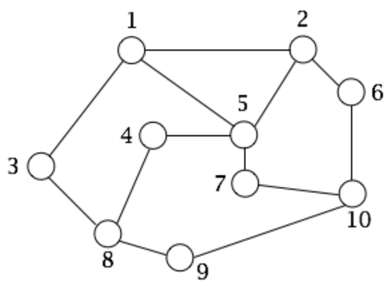
- V is the (finite) set of vertices.
- The set of edges E represents the relations between all pairs of vertices.

There exist a large variety of graphs:
directed graphs, weighted graphs, networks, multiple graphs, hypergraphs, etc..

How to represent graphs?

- Picturally
- Algebraically (Adjacency matrices)

An example.

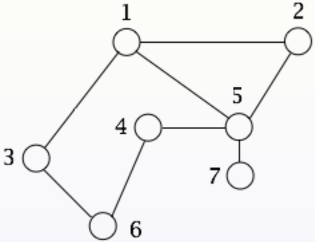


Adjacency matrices

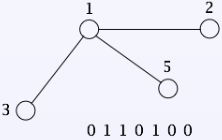
In the previous graph:
vertex 1 is linked with 2, 3 and 5
vertex 2 is linked with 1, 5 and 6
vertex 3 is linked with 1 and 8
...

These relations are represented by a matrix as follows:
$A_{i,j} = 1$ if vertex i is linked with vertex j $(i \neq j)$
$A_{i,j} = 0$ otherwise

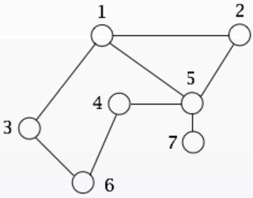
Adjacency matrices



For each vertex, it represents its connected vertices.
Example for vertex 1.

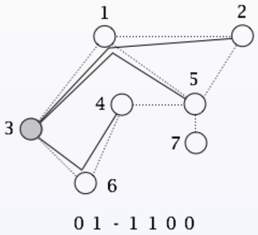


The whole adjacency matrix



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Let compute the distance-2 graph and its adjacency relation.
Example for vertex 3.



Moreover, 3 is also linked twice by paths of length 2.

Interest of the Algebraic approach

Compute A^2.

$$A^2 = \begin{bmatrix} 3 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Interest of the Algebraic approach

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The element $A^2_{i,j}$ in row i and column j gives the number of paths of length exactly equal to 2 between i and j.

Graph Characteristics

Several notions tell us about the graph structures:

- Degree
- Diameter
- Chromatic index (detailed in the next lecture)

The last characteristic is hard to compute while both others are easy to obtain (in polynomial time).

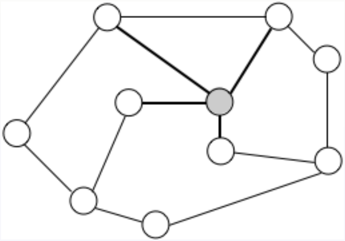
Degree

Definition

The degree of vertex x is the number of vertices adjacent to x.
Denoted by $\delta(x)$ for $x \in V$

- The maximum (resp. average) degree of a graph is the maximum (resp. average) of the degree of its vertices.
- A graph where all the degrees are equal is *regular*.

Example



The degree of the shaded vertex is 4.

Path is graph

Definitions

- A path is sequence of edges such that two adjacent edges share a vertex.
- The length of the path is the number of edges of the sequence.

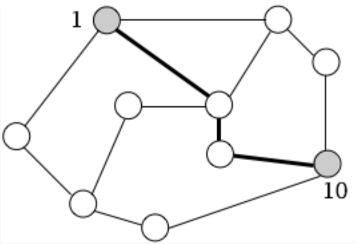
The distance can be extended easily to weighted graphs by summing up all the weights in the sequence.

Diameter

We define a *natural distance* between two vertices as the minimum number of edges to cross for connecting both vertices.

Definition

- maximum distance between any pairs of vertices
  It is denoted by D



The length of the path between the two shaded vertices is 3.
Notice that the Diameter of the graph is also 3.

Graph connectivity

We introduce the relation x **is connected to** y if there exists a path between x and y.

Property

The connectivity is an equivalence relation.

Proof

Easy by applying the definition.

- **Reflexivity:** Obviously, a vertex is connected to itself
- **Symmetry:** A path from x to y is also a path from y to x
- **Transitivitty:** If there exists a pathfrom x to y and a path from y to z, then there is a path from x to z.

An equivalence relation induces *connected components*, i.e. classes where the vertices are two by two connected.

Extension to directed graphs

Here, a path takes into account the direction of the *arcs*.

Strongly connected relation

There exists a path between x and y and a path between y and x.

Link with order relations

Directed acyclic graphs

Proposition.

DAG is the representation of a partial order relation.
Topological order.

- Show that in a DAG there is a vertex with no predecessor.

Proof

What kind of proof could be used?

Proof

What kind of proof could be used?

Using an extremal property

There exists a longest path.

- Focus on the first vertex of this path.
- It has no predecessor, because
  - The path is extremal
  - There is no cycle

An important/useful proposition

Statement

In any undirected graph, the number of the vertices with odd degrees is even.

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Proof

- $\sum_{x \in V} \delta(x) = 2 \cdot |E|$

  Thus, $\sum_{x \in V} \delta(x)$ is even

An important/useful proposition

Statement

In any undirected graph, the number of the vertices with odd degrees is even.

Proof

- $\sum_{x \in V} \delta(x) = 2 \cdot |E|$

  Thus, $\sum_{x \in V} \delta(x)$ is even
- $\sum_{x \in V} \delta(x) = \sum_{x \in Vodd} \delta(x) + \sum_{x \in Veven} \delta(x)$

  obviously, $\sum_{x \in Veven} \delta(x)$ is even,

  thus, $\sum_{x \in Vodd} \delta(x)$ is even

  Necessarily, their number is even.

Chromatic index

Two adjacent vertices are colored in a different color.

Definition

minimum number of colors for coloring the graph.

Related properties will be detailed in the lecture next week.