UE Mathematics for Computer Science

I. Miscellaneous Exercises

I.1. Choosing a team

You want to choose a team of m people from a pool of n people for your startup company, and from these people you want to choose k to be the team managers. You took the *Mathematics for omputer Science course*, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your manager, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n \quad k}{m \quad k}$$

Before doing the reasonable thing, dump on your manager, you decide to check his answer against yours.

1. Start by giving an algebraic proof that your manager's formula agrees with yours.

2. Now give a combinatorial argument proving this same fact.

I.2. curious decomposition

Now try the following, more interesting theorem:

$$n2^{n \quad 1} = \sum_{k=}^{n} k\binom{n}{k}$$

1. Start with a combinatorial argument. Hint: let $\mathcal{S}$ be the set of all sequences in $\{0, 1, \star\}^n$ containing exactly one $\star$.
2. How would you prove it algebraically?

I.3. Covering

Let $\mathcal{E}$ a set of n elements. 2-covering is a couple subsets (A, B) of $\mathcal{E}$ such that $A \cup B = \mathcal{E}$. Compute the number of 2-covering.

I.4. No adjacency

There are 20 books arranged in a row on a shelf.

1. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15bit sequences with exactly 6 ones.
2. How many ways are there to select 6 books so that no two adjacent books are selected?

I.5. **Combinatorial identity**

Prove the following theorem

$$\sum_{i=}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

1. using a combinatorial argument;
2. using induction.

I.6. **Sums of i^k**

I.6.a. Prove by a combinatorial argument that

$$\sum_{i=1}^{n} i^k \text{ is a polynomial with degree } k+1$$

I.6.b. Compute

$$\sum_{i=k}^{n} \binom{i}{k}$$

I.6.c. Propose a method to compute

$$\sum_{i=1}^{n} i^k$$

I.7. Decomposition

I.7.a. Prove the equality using a combinatorial argument

$$\binom{n}{k} = \binom{n \quad 1}{k} + \binom{n \quad 1}{k \quad 1}$$

I.7.b. Prove the equality using a combinatorial argument

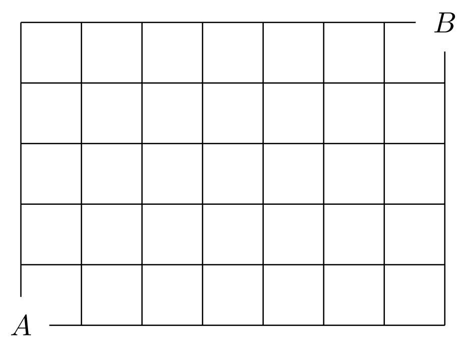
$$\binom{n}{k} = \binom{n \quad 2}{k} + 2\binom{n \quad 2}{k \quad 1} + \binom{n \quad 2}{k \quad 2}$$

I.7.c. Propose a generalization of these results

I.8. Prove the following identity:

$$\sum_{k= \quad k \text{ even}} \binom{n}{k} = \sum_{k= \quad k \text{ odd}} \binom{n}{k} \text{ for all } n \geqslant 1$$

I.9. In a rectangle grid $m \times n$ compute the number of shortest paths from $A = (0,0)$ to $B = (m,n)$.



I.10. Cycles in permutations

Consider $\mathcal{S}_n$ the set of all permutations on n objects.

- I.10.a. Prove that a permutation could be uniquely decomposed in a set of disjoint cycles.
- I.10.b. Recall the number of permutations in $\mathcal{S}_n$ with only one cycle and prove it.
- I.10.c. Compute the number of permutations having exactly two cycles.
- I.10.d. Generalize the computation method with an arbitrary number of cycles

I.11. Programming

- I.11.a. Choose a data structure to represent sets and subsets
- I.11.b. Design an algorithm that enumerate all the subsets of size k
- I.11.c. Write a program that enumerates all the subsets of size k from a set S
- I.11.d. Prove this algorithm
- I.11.e. Evaluate the complexity of this algorithm
- I.11.f. How could the program be modified to enumerate subsets with size less ( ) than k ?
- I.11.g. How could the program be modified to enumerate subsets with size k an an even number of elements ?

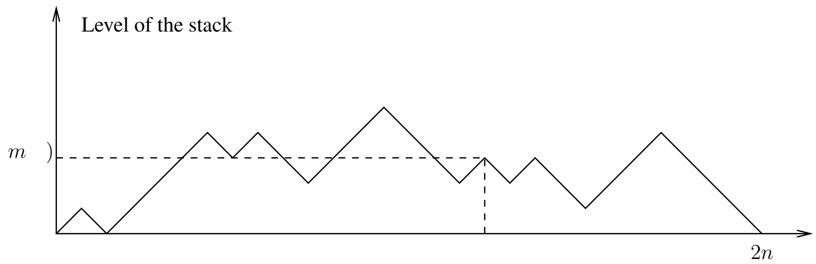
I.12. Programming (2)

- I.12.a. Write a program that computes $\binom{n}{k}$ from the algebraic definition
- I.12.b. Write a recursive program that computes $\binom{n}{k}$ from the Pascal's triangle
- I.12.c. Write a program that computes $\binom{n}{k}$ from the Pascal's triangle and using dynamic programming
- I.12.d. nalyze these algorithms (complexity efficiency,...

II. Counting Stack Orderings

Consider a stack with the two primitives `pus` and `pop` n execution of a program consists in n operations `pus` and n `pop`which could be interleaved.

The execution is represented the function $m(k)$ that gives the level of the stack after k operations. We denote this function as a *mountain*,



Corresponds to the sequence `pus` , `pop`, `pus` , `pus` , `pus` , `pus` , `pop`, `pus` , `pop`, `pop`,...

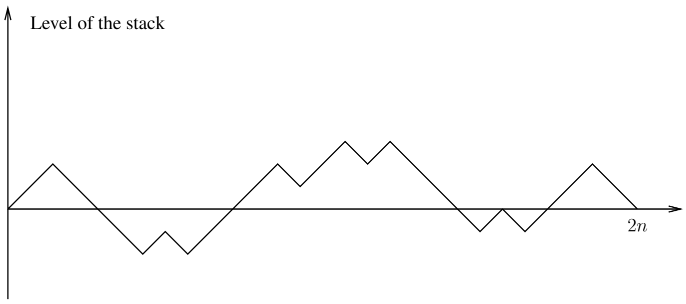
The curve $m(k)$ is always above the x axis (if the stack is empty at the beginning, the number of `pus` must be always greater or equal to the number of `pop`until the end of execution, finishing with an empty stack).

Denote by M_n the number of *mountains* with n `pus` (up-stroke) and n `pop`(down-stroke) operations and set $M = 1$.

II.1. **Small n cases**

For $n = 1, 2, 3$ give the possible *mountains* and deduce M_1, M_2, M_3.

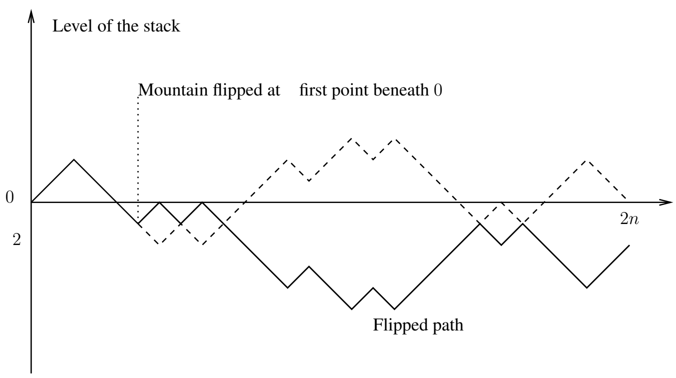
n extended *mountain* of length $2n$ with n up-strockes and n down-strockes could be below the x axis :



Extended mountains could not be a correct execution of a sequence of `pus` and `pop`, because sometimes it allows `pop`on an empty stack.

II.2. **Extended *mountains*** Compute the number of extended mountains with length $2n$.

The flip operation consists in exchanging all the slopes after the first passage below 0:



II.3. **Flipped mountains**

Show that the set of extended mountains that goes below the x axis is in bijection with the set of extended mountains with $n \quad 1$ up-strokes and $n+1$ down-strokes.

II.4. **Computation**

Prove that

$$M_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n}{(n+1)\ n} \tag{1}$$

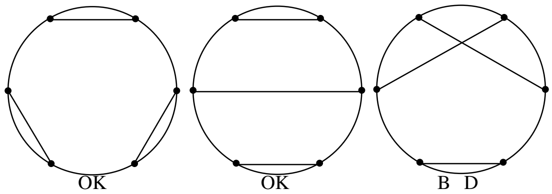
II.5. **Recurrence relation** Show directly on mountain diagrams that the M_n numbers satisfy the recurrence equation:

$$M_n = M \ M_{n \ 1} + M_1 M_{n \ 2} + \cdots + M_{n \ 1} M \tag{2}$$

III. Examples

III.1. **Shake hands**

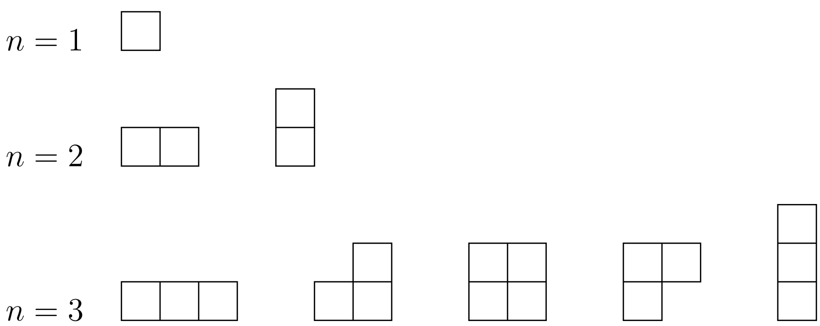
Some shaking hands configurations with $n = 3$



Suppose that $2n$ persons are seated around a table, how many ways could they shake hands without crossing ?

III.2. **Circuits**

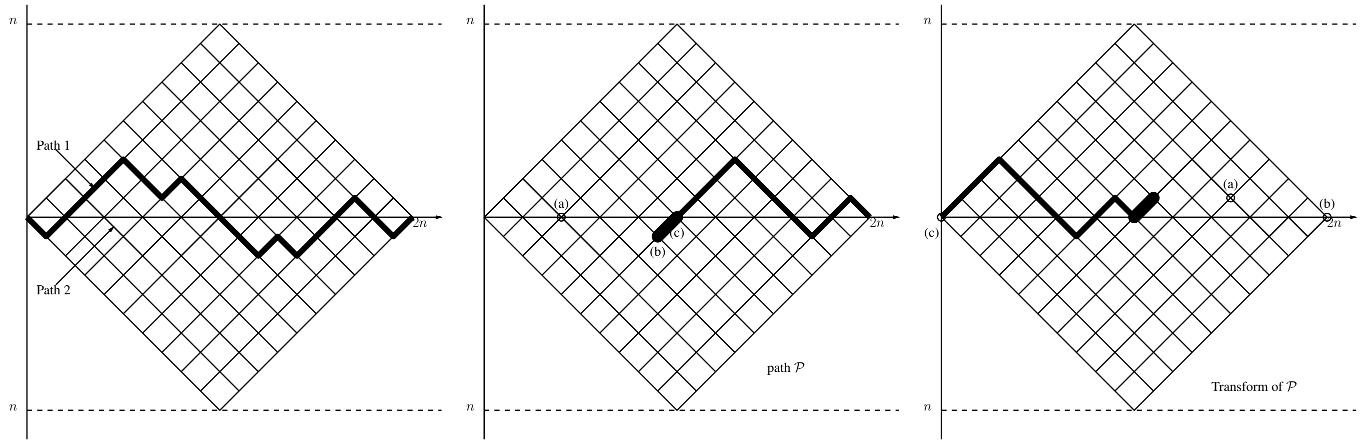
Circuits with perimeter $2n + 2$



How many shapes of circuits could be done with $2n + 2$ unit segments ?

IV. Paths in a diamond *xam 2018*

Consider paths of length $2n$ in a diamond of size $n \times n$ from $(0, 0)$ to $(2n, 0)$. Two paths are given in the Figure 1a.



(a) Two path in a diamond (b) path $\mathcal{P}$ (c) Transform of path $\mathcal{P}$

Figure 1: Paths in a 10×10 diamond

IV.1. Give all such paths for the 3×3 diamond.

IV.2. Compute P_n the number of paths in a $n \times n$ diamond (give the proof).

For a path $\mathcal{P}$ we define the below number $B(\mathcal{P})$ counting the number of steps that are below the x-axis and going down.

IV.3. What are the possible values of $B(\mathcal{P})$? Compute $B(\mathcal{P})$ for all the paths in the 3×3 diamond and for the two paths given in the Figure 1a.

Consider now the following transformation T of a path illustrated in the figures 1b and 1c. Take the first point (a) when the path cuts the x-axis, take the point (c) when the path hits again the x-axis for the first time by the step $(b) \to (c)$. Then rearrange the path by shifting the last part of the path up to 0, put the step $(b) \to (c)$ and finish by the first part of the path. Of course this transform applies on paths with positive below number

IV.4. Prove and compute the inverse transform.

IV.5. Prove that, if $B(\mathcal{P}) \geqslant 1$ we have $B(T(\mathcal{P})) = B(\mathcal{P})$ 1. nd deduce that the set of paths with below number p is in bijection with the set of paths with below number p 1.

IV.6. What are the paths satisfying $B(\mathcal{P}) = 0$? How many are they ?

IV.7. (optional) What is the name of the number of paths such that $B(\mathcal{P}) = 0$?

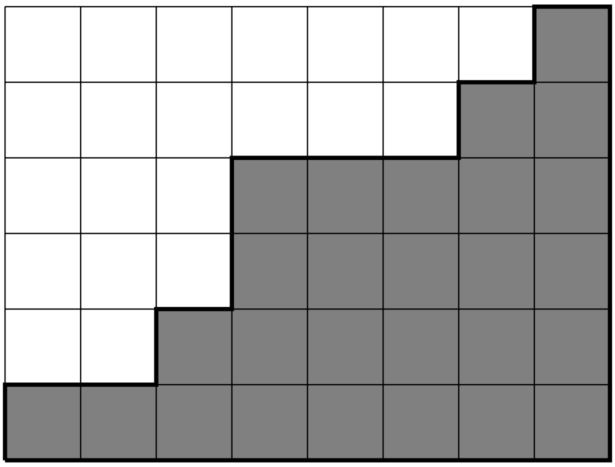
IV.8. Prove that the set of paths that never go below the x-axis is in bijection with the set of complete[1] binary trees.

IV.9. Propose an algorithm that generate uniformly a complete binary tree and compute its complexity.

[1] Complete means that each node has either or 2 child.

V. Design of stairs

Consider a typical stair build with 1×1 squares



The length of this stair is 8, its height is 6 and its size is 27 small squares.

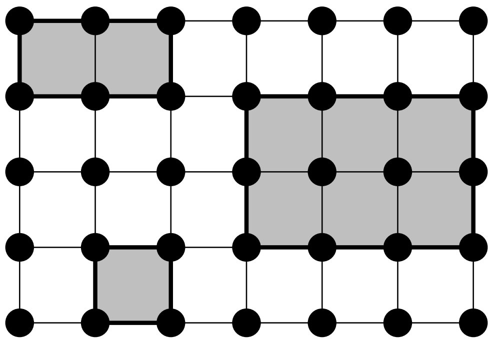
V.1. Compute the number of stairs with length 8 and height 6

V.2. Compute the number of stairs with size 27

V.3. nalyse your methodology to solve these problems.

VI. Rectangles and Squares in Grids **Mid-term 2023**

Consider a 7×5 grid. In the following figure we have 3 typical rectangles (squares are particular cases of rectangles). The corners of the rectangles are points of the grid and the sides are parallel with the x-axis or the y-axis, a single point of the grid is not considered as a rectangle.

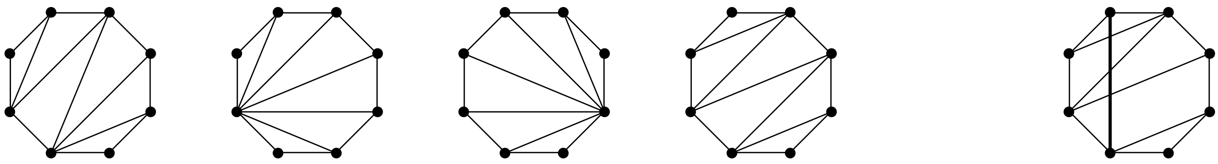


VI.1. Compute the number of rectangles in the 7×5 grid. Justify carefully the method used.

VI.2. Is it possible to apply the method to compute the number of squares in the 7×5 grid. Compute the number of squares.

VII. Problem: cut a polygon — Mid-term 2022

Consider a regular octagon, 8 corners, 8 sides. We want to cut this octagon in triangles, cuts should be done between corners and two cuts should not cross. Here are four examples of cuts and a non admissible cut (cuts intersect).



Non admissible

VII.1. Propose a method to evaluate the number of cuts for the octagon and compute this number. Comment the problem (give informations on the problem that look interesting for you).

VIII. Partition of integers — Mid-term 2021

Consider the problem of allocating n **unlabelled** balls in k **unlabelled** non-empty urns. The aim of the problem is to count the number of such allocations.

nother problem, called integer partition, is to count the number of ways to decompose an integer in a sum of integer. Of course $3+1+1$ is the same decomposition of 5 as $1+3+1$, but is different of $2+2+1$.

VIII.1. State the problem, fix the notations and establish the link between the two problems.

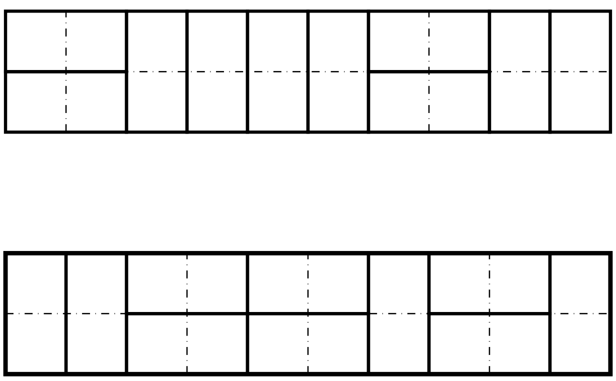
VIII.2. Compute the number of allocations for $k = 2$

VIII.3. For $n = 10$ and $k = 4$ compute the number of allocations.

VIII.4. Establish a recurrence equation to compute the number of allocations for any n and k. The limit conditions should be given precisely.

IX. Problem on friezes — Exam 2023

We consider tiles with size 2×1. frieze with height 2 , consists in a succession of tiles put horizontally or vertically without holes. Here are two examples of friezes of length 10.



The aim of this problem is to build a random frieze of 2×1 tiles of length 20.

IX.1. Propose an algorithm that generates a random frieze.

omment: detail and justify the approach, give a proof of the algorithm, evaluate its complexity, give examples, provide all the elements to convince on the quality of the proposed solution