On a batch scheduling problem with compatibility and cardinality constraints

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1 Introduction

In several communication systems messages are to be transmitted in a single hop from senders to receivers through direct connections established by an underlying switching network. In such a system, a sender (resp. receiver) cannot send (resp. receive) more than one messages at a time, while the transmission of messages between different senders and receivers can take place simultaneously. Usually, a constraint to the number of the simultaneous transmissions exists. The scheduler of such a system establishes successive configurations of the switching network, each one routing a non-conflicting subset of the messages from senders to receivers. Given the transmission time of each message, the transmission time of each configuration equals to the longer message transmitted. The aim of the scheduler is to find a sequence of configurations such that all the messages to be finally transmitted and the total transmission time to be minimized.

This problem is known as Time Slot Scheduling and it is equivalent to a parallel batch scheduling problem with (in)compatibilities between jobs. In this batch scheduling problem jobs correspond to the edges of an undirected *graph* and adjacent jobs cannot be scheduled in the same batch. This problem is denoted as $1 \mid p-batch,\ k<n,\ graph \mid C_{max}$, where k is a given bound on the batches cardinality, and it is also equivalent to the following weighted edge coloring problem: given an undirected weighted graph $G=(V,E)$ and an integer $k \leq |E|$, we ask for a partition $M=\{M_1, M_2, \ldots, M_s\}$ of the edges of G into matchings, each one of weight $w(M_i)=\max\{w(e)|e \in M_i\}$, such that $|M_i| \leq k$ and $w(M)=\sum_{i=1}^{s} w(M_i)$ is minimized. In the following we shall refer to this problem as $k-$Batch Scheduling-Edge Coloring (k-BSEC) problem. By BSEC we refer to the problem without a bound k on the matchings cardinality.

A body of the work carried out last years is concentrated in the preemptive variants of the BSEC [3] and k-BSEC [1] problems, as well as in the non-preemptive BSEC problem [6, 4]. Moreover, the analogous Batch Scheduling-Vertex Coloring problem has been also studied (see e.g. [5, 4, 7]).

In this paper we study the non-preemptive k-BSEC problem. Since the BSEC problem is NP-hard [6, 4], it follows directly that the k-BSEC problem is also NP-hard. In the following we present a 3-approximation algorithm for the k-BSEC problem as well as a polynomial algorithm for the 2-BSEC problem.

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2 An approximation algorithm for the k-BSEC problem

A 2-approximation greedy algorithm for the BSEC problem has been presented in [6]. We generalize this algorithm for the k-BSEC problem.

Algorithm 1

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1. Sort the edges of G in non-increasing order of their weight
2. Using this order find successive maximal matchings M_1, M_2, ... M_s
    such that |M_i| ≤ k,  1 ≤ i ≤ s
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Lemma 2 *Let $w_1^* \geq w_2^* \geq \ldots \geq w_{s^*}^*$ be the weights of the matchings of an optimal solution. For the i-th matching M_i found by Algorithm 1 it holds that $w(M_i) \leq w^*_{\lceil \frac{i}{3} \rceil}$*

Proof: (sketch) Let $e = (u, v)$ be the heaviest edge in the i-th matching. This edge is not selected by the algorithm in any previous matching M_j, $1 \leq j < i$, because either it has a conflict (i.e. a common endpoint) with a heavier edge in M_j or M_j has already exactly k edges. Let x and y be the number of matchings for which a conflict for the edge e occurred due to its endpoints u and v, respectively. Let z be the number of matchings for which e is not selected because they were full. Then, it holds that $x + y + z \geq i - 1$. We distinguish between two cases w.r.t. the magnitude of z ($z \geq \left\lceil \frac{i-1}{3} \right\rceil$ and $z < \left\lceil \frac{i-1}{3} \right\rceil$) and in both cases we prove that $w(M_i) \leq w^*_{\lceil \frac{i}{3} \rceil}$. ■

Summing up the inequalities of Lemma 2 we obtain

Theorem 3 *Algorithm 1 achieves a 3-approximation ratio for the k-BSEC problem.*

3 A polynomial algorithm for the 2-BSEC problem

Since $k = 2$ any solution in this case consists of matchings of only one or two edges. Let A be the set of edges in the one-edge matchings of a solution. Note that for such a set A it holds that the number $|E| - |A|$ is an even one. By Δ we denote the maximum vertex degree of G. Let E_v be the subset of edges adjacent to a vertex $v \in V$. The set A of an optimal solution either is a subset of E_v for some $v \in V$ or it forms a triangle; otherwise two of A's edges can be combined into a single matching. Next lemma is based on this fact.

Lemma 4 *For any vertex $v \in V$ there are $O(\Delta^5)$ different subsets A, including edges adjacent to v, that should be examined in order to find an optimal solution to the 2-BSEC problem.*

Proof: (sketch) If all edges in E_v have distinct weights and an edge $e \in E_v$ of weight $w(e)$ does not belong in A, then, by a swapping argument, it follows that at most two other edges of E_v of weights greater than $w(e)$ can belong in A. If there are r edges in E_v with the same weight and more than two of them are in A then, using again a swapping argument, it follows that does not matter which of them is the third, forth, ..., $(r-1)$-th. The different subsets of E_v satisfying the above restrictions are $O(\Delta^5)$, which reduces to $O(\Delta^3)$ when all edges of E_v have distinct weights. Moreover, a vertex v can belong in at most $O(\Delta^2)$ triangles. ■

Lemma 4 is used in the next optimal algorithm for the 2-BSEC problem.

Algorithm 5

1. For each vertex $v \in V$
2. For all $O(\Delta^5)$ different sets A of one-edge matchings
3. Create the line graph $L = (E - A, F)$ of the subgraph of G induced by its $E - A$ edges
4. For each edge $(x, y) \in F$ set $w(x, y) = \max\{w(x), w(y)\}$
5. If L has a perfect matching
6. Find the minimum weighted perfect matching in L (two matched vertices in L correspond to a two-edges matching in G)
7. Create a solution consisted of:
   - The matchings found in step 6
   - The A one-edge matchings
8. Return the best solution found

Algorithm 5 calls $O(|V| \cdot \Delta^5)$ times the procedure of finding the minimum weighted perfect matching. Such a matching can be found in polynomial time (see e.g. [2]).

Theorem 6 *Algorithm 5 finds an optimal solution to the* 2-BSEC *problem in polynomial time.*

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