Fundamental Computer Science
Analysis of Vertex Cover

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Agenda

- A detailed example: Vertex Cover

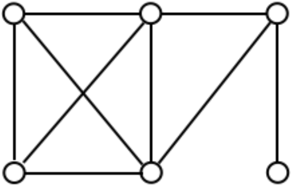
Presentation of the Vertex Cover problem

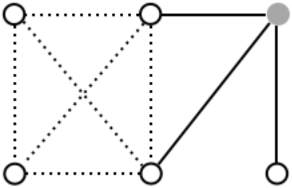
Covering the vertices of a graph by some of its edges.

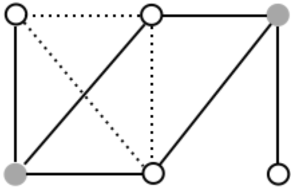
- **VC**
- **Input.** A graph $G = (V, E)$ given for instance by its adjacency matrix and an integer Q
- **Question.** Is there a set V' with at most Q vertices that are covering G?

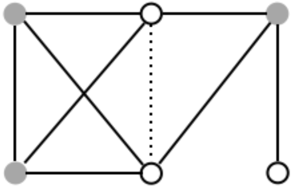
This means that each edge of G has at least one of its extremities in V'.

Example

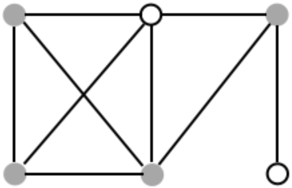








Final covering with 4 sommets



VC is in NP-COMPLETE

We first verify that VC $\in \mathcal{NP}$.

Verifier

- generate non-deterministically a set of vertices
- verify that this set is a covering

The verifier is non-deterministic polynomial.

Reduction

We transform a (positive) instance (positive) of 3SAT into a (positive) instance of VC.

n variables p_i

$C_1 \wedge C_2 \wedge \cdots \wedge C_m$
where $C_i = x_{i,1} \vee x_{i,2} \vee x_{i,3}$
where $x_{i,j}$ is a literal on the $\{p_1, p_2, \cdots, p_n\}$

Construction of the associated graph (the vertices)

- A pair of vertices between each propositional variable p_i and $\neg p_i$
- A triplet of vertices for each clause C_i

The number of vertices is thus equal to $2n + 3m$

Construction of the associated graph (the edges)

- An edge between each pair p_i and $\neg p_i$
- An edge between each of the three vertices of the triangles C_i
- An edge between each $x_{i,j}$ and the vertex p or $\neg p$ depending of the literal

The number of edges is thus equal to $3m + 3m + n$

Construction of the associated graph (the constant Q)

$Q = 2m + n$

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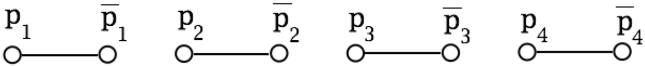
$Q = 2m + n$

Of course, this transformation is polynomial.

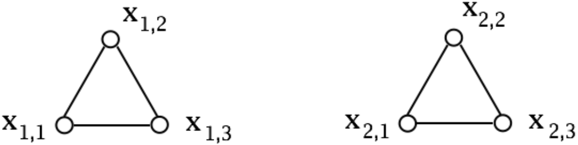
Example (with 4 variables)

$(p_2 \vee \neg p_1 \vee p_4) \wedge (\neg p_3 \vee \neg p_2 \vee \neg p_4)$



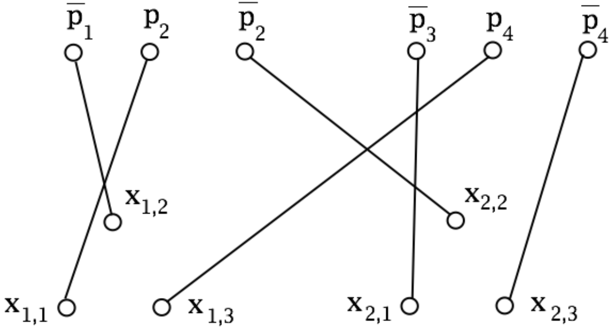
Example (2 triangles)

$(p_2 \vee \neg p_1 \vee p_4) \wedge (\neg p_3 \vee \neg p_2 \vee \neg p_4)$



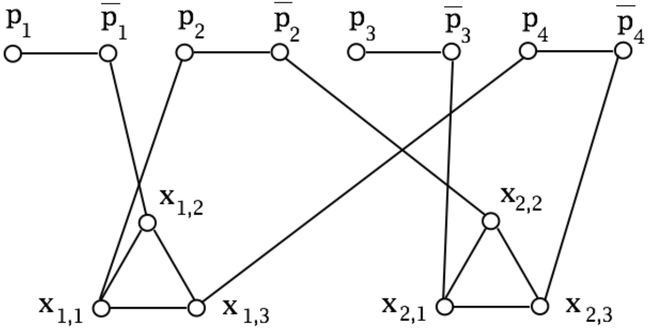
Example (associations between x and p)

$(p_2 \vee \neg p_1 \vee p_4) \wedge (\neg p_3 \vee \neg p_2 \vee \neg p_4)$



The final graph

$(p_2 \vee \neg p_1 \vee p_4) \wedge (\neg p_3 \vee \neg p_2 \vee \neg p_4)$



First, let remark that the previous transformation is polynomial.

We prove now that

3SAT is satisfiable iff the transformed graph has a covering at most $2m + n$.

Proof ($\Rightarrow$)

- Let consider a positive instance of 3SAT
- There exists an interpretation function γ that affects $TRUE$ to all the clauses C_i.
  The covering is derived from this function, it contains the following vertices.
  - The vertices p_i for which γ affects $TRUE$ and the $\neg p_i$ for which it affects $FALSE$.
  - 2 vertices for each triangle, chosen such as γ affects $TRUE$ to the non-chosen vertex[1].

  The size of this covering is exactly $n + 2m$.

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  - 2 vertices for each triangle, chosen such as γ affects $TRUE$ to the non-chosen vertex[1].

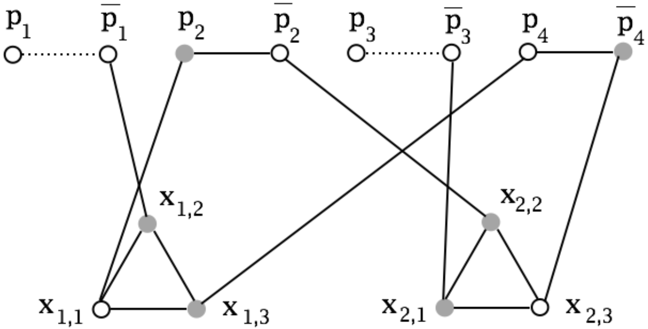
  The size of this covering is exactly $n + 2m$.
- We should also verify that all the edges are well covered.
  This is easy by considering successively each type of edge.

[1] that is always possible since as all the clauses are $TRUE$, the function affects at least one literal to $TRUE$ in each clause

A solution of VC on the previous example

$(p_2 \vee \neg p_1 \vee p_4) \wedge (\neg p_3 \vee \neg p_2 \vee \neg p_4)$

$(p_2 = 1, \neg p_4 = 1)$ is solution (whatever the value of p_1 and p_3).



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- A covering of size $2m + n$ contains necessarily a vertex of each pair $(p_i, \neg p_i)$ and two vertices in each triangle.
  It could not have more because in this case, the size will be more than $2m + n$.
- The interpretation function is the one that affects $TRUE$ to p_i if it is in the covering set and $FALSE$ if $\neg p_i$ is not.
  The existence of this covering implies that the corresponding 3SAT formula is $TRUE$ since one of the $(p_i, \neg p_i)$ is covered for each triangle (clause).