Fundamental Computer Science
Lecture 3: first steps in complexity
Reductions

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Agenda

- Reduction
- Goal: to classify the problems in complexity classes
- A focus on randomized algorithms
- (if enough time): The class NP-complete (Cook's Theorem)

Reductions

Definition

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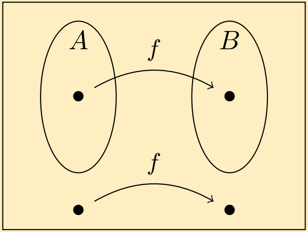
A language A is **polynomial time reducible** to language B, denoted $A \leq_{\mathrm{P}} B$, if there is a polynomial time computable function $f : \Sigma^* \to \Sigma^*$, where for every input w, it holds that

$$w \in A \Longleftrightarrow f(w) \in B$$

This function f is called a **polynomial time reduction** from A to B.



Reductions

Theorem

If $A \leq_{\mathrm{P}} B$ and $B \in \mathcal{P}$, then $A \in \mathcal{P}$.

Proof:

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- f: a polynomial time reduction from A to B
- Create a polynomially bounded Turing Machine M' deciding A

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- Create a polynomially bounded Turing Machine M' deciding A

  M' = "On input w:

  1. Compute $f(w)$.
  2. Run M on $f(w)$ and output whatever M outputs."

A first straightforward example

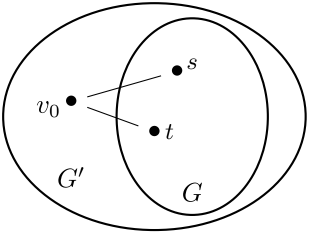
$\mathrm{HPATH} = \{\langle G, s, t\rangle \mid G$ is a graph with a Hamiltonian path from s to $t\}$

$\mathrm{HCYCLE} = \{\langle G\rangle \mid G$ is a graph with a Hamiltonian cycle$\}$

Show that HPATH is polynomial time reducible to HCYCLE.

Solution:

- input of HPATH: a graph $G = (V, E)$ and two vertices $s, t \in V$
- create an instance of HCYCLE
  - $G' = (V', E')$ where $V' = V \cup \{v_0\}$ and $E' = E \cup \{(v_0, s), (v_0, t)\}$



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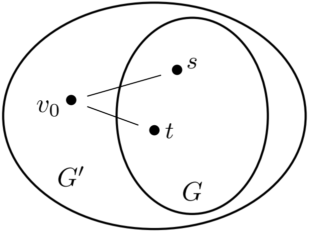

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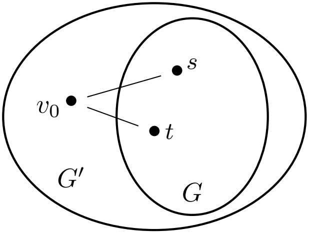

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We are not done!!!

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There is a Hamiltonian Path from s to t in G if and only if there is a Hamiltonian Cycle in G'

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- there are only two edges incident to v_0: (s, v_0) and (t, v_0)
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- Hamiltonian Cycle in G': $t \to v_0 \to s \to \ldots \to t$
- there is a Hamiltonian Path from s to t in G $\square$

Steps of a reduction

Reduction from A to B

1. transform an instance I_{A} of A to an instance I_{B} of B
2. show that the reduction is of polynomial size
3. prove that:
   "there is a solution for the problem A on the instance I_{A}
   if and only if
   there is a solution for the problem B on the instance I_{B}"

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Comments

- usually the one direction is trivial (due to the transformation)
- $|I_{\mathrm{B}}|$ is polynomially bounded by $|I_{\mathrm{A}}|$

A slightly modified problem

Let us extend our previous example:

$\mathrm{HPATH} = \{\langle G \rangle \mid G$ is a graph with a Hamiltonian path $\}$

- HPATH
- **Instance:** A graph $G = (V, E)$
- **Question** Is there an hamiltonian path in G?

We want to show that this problem reduces to $HCYCLE$.

Example (cont'd)

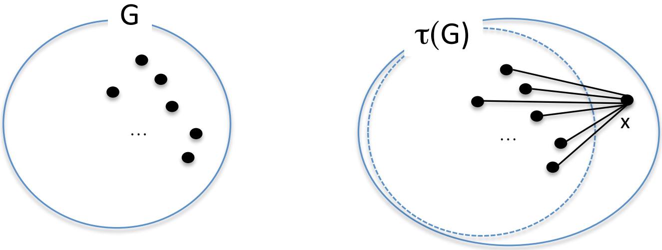
Let us consider an instance I_{HPATH}, that is a graph G.

Example (cont'd)

Let us consider an instance I_{HPATH}, that is a graph G.

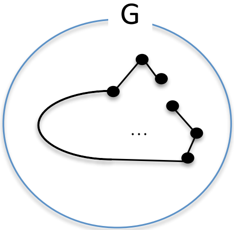
- We build a particular instance $\tau(G) = G'$ of $HCYCLE$ by adding a new vertex x that is linked with all the other vertices of G:
- $G' = (V', E')$ where $V' = V \cup \{x\}$ and $E' = E \cup \{x, y\}\ \forall y \in V$

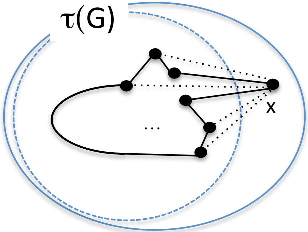
Principle of the reduction from $HPATH$ to $HCYCLE$



Principle (cont'd)

A Hamiltonian path in G (left) leads to a cycle in $\tau(G)$ (right)





Proof

- The transformation τ is obviously polynomial.
- Let us **show that it is a reduction**:
  G has an hamiltonian path if and only if $\tau(G)$ has an hamiltonian cycle.

- $(\Rightarrow)$
  If G has an hamiltonian path (called φ), then, the cycle $x \to \varphi \to x$ is hamiltonian in $\tau(G)$.
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  If G' has an hamiltonian cycle, its sub-graph without x, G, has an hamiltonian path.
  - consider a Hamiltonian Cycle in G'

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  - the hamiltonian Cycle in G' is $t \to x \to s \to \ldots \to t$
  - there is a Hamiltonian Path from s to t in G $\square$

Exercise

It is also possible to establish the following reduction:
HCYCLE $\leq_{\mathrm{P}}$ HPATH.

This result is not immediat, even if it *is/seems* easy to extract a path from a cycle...

What is the problem here?

Exercise

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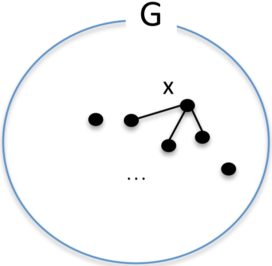
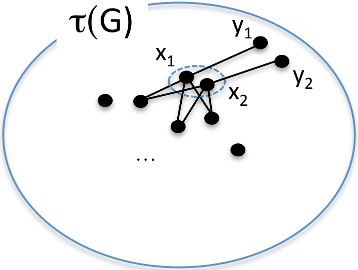
What is the problem here?

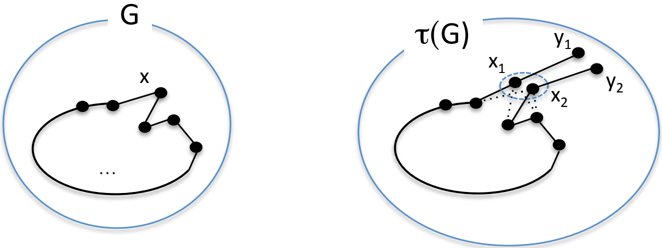
We can not have a characterization of the hamiltonian path (and particularly of its extremities...)

Principle of the reduction from HCYCLE to HPATH

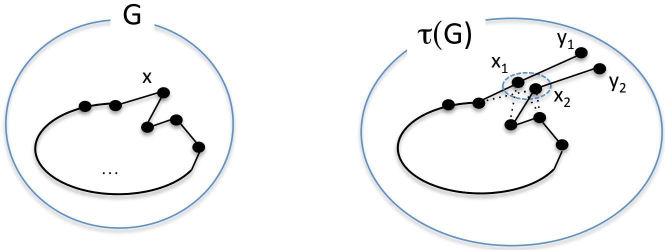
τ transforms an instance G of HCYCLE to an instance $\tau(G)$ of HPATH:







- We duplicate one vertex (any one) x of G in (x_1, x_2)



- We duplicate one vertex (any one) x of G in (x_1, x_2)
- We link x_1 and x_2 respectively to two new vertices y_1 and y_2 as depicted in the figure.

Analysis

This transformation is polynomial.

Show that it is a reduction:
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- $(\Rightarrow)$
  If G has an hamiltonian cycle, the path starting at $y_1 \rightarrow x_1$, joining the cycle until reaching the neighbor of x_1, then, $x_2 \rightarrow y_2$ is an hamiltonian path in $\tau(G)$.
- $(\Leftarrow)$
  - If there is a hamiltonian path φ in $\tau(G)$, it is necessarily like
    $y_1 \rightarrow x_1 \rightarrow \psi \rightarrow x_2 \rightarrow y_2$
    since there is no other choice for y_1 and y_2
  - Then, $x \psi x$ is an hamiltonian cycle in G.

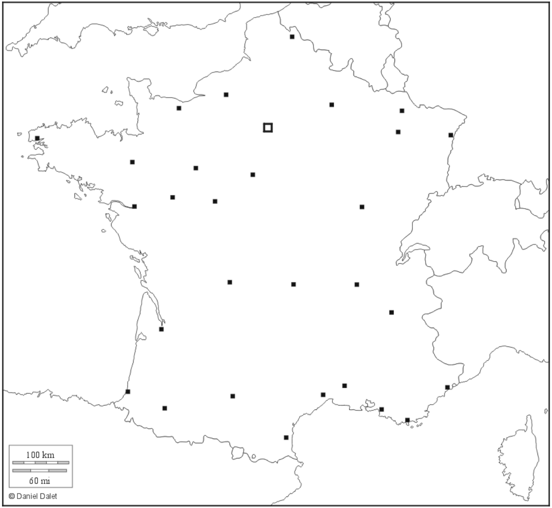
Exercise: TSP

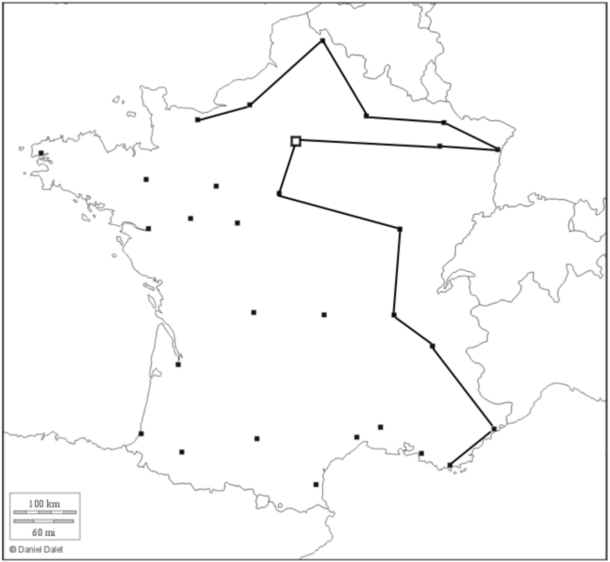
We consider now the decision version of the Travel Saleswoman Person (D-TSP).

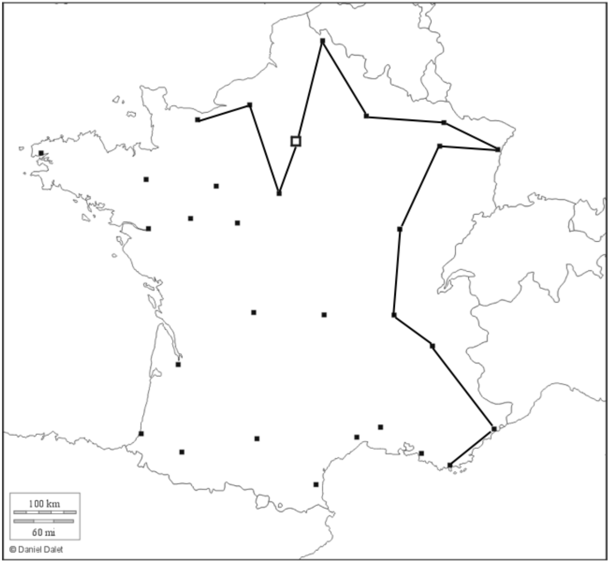
- D-TSP
- **Instance.** a set V of n cities with the distance matrix $(d_{i,j})$ and an integer k.
- **Question.** is there an itinerary of length at most k passing through each city exactly once?

Show HCYCLE $\leq_{\mathrm{P}}$ D-TSP

Illustration on an instance




100 km
60 mi
© Daniel Dalet


100 km
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Principle of the reduction

The set of the cities corresponds to the vertices of the graph G, the distances are given by the particular matrix $d_{i,j} = 1$ if i and j are linked, 2 otherwise.
The constant k is equal to n (number of cities).

This transformation is polynomial, show that it is a reduction.