Fundamental Computer Science
Lecture 3: first steps in complexity
Complexity classes

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Summary of the previous lecture

- Turing Machines
  - universal computational model
  - non-determinism
    decide the same languages as the deterministic TM... but not using the same number of steps
  - all variants of the model are equivalent w.r.t. *decidability*
  - the RAM is a good trade-off

Agenda

- A focus on decidability
- Classifying the problems in complexity classes
  - time complexity:
  - space complexity

Focus on *decidable* languages (that correspond to *solvable* problems)

Time complexity class

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We define the **time complexity class**

$\mathrm{TIME}(f(n)) = \{L \mid L$ is a language *decided* by a Turing Machine in $O(f(n))$ time, where n is the size of the input$\}$

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$L \in \mathrm{TIME}(n^2)$

The length of the input is $n = 2k$

Example (2)

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An enhanced Turing Machine:

$M_2 =$ "On input w:

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$L \in \mathrm{TIME}(n \log_2 n)$

The class $\mathcal{P}$

A Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$ is called **polynomially bounded** if there is a polynomial p and for any input w there is no configuration C such that $(s, \sqcup w) \vdash_M^{p(|w|)} C$.

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$\mathcal{P}$ is the class of *polynomially decidable* languages.

$$\mathcal{P} = \bigcup_k \mathrm{TIME}(n^k)$$

Recall: languages versus problems

- Decision problem: a problem whose answer is yes/no.
- Example
  - PATH
  - Input: Given a graph $G = (V, E)$ and two nodes $s, t \in V$
  - Question: is there a path from s to t?

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  $\mathrm{PATH} = \{\langle G, s, t\rangle \mid G$ is a graph that has a path from s to $t\}$

  - $\langle G, s, t\rangle$ is the input string
  - $|\langle G, s, t\rangle|$ is the size of the input[1]

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- $\mathrm{PATH} \in \mathcal{P}$?
  - **Yes** (i.e., Breadth First Search in $O(|V| + |E|)$)

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Enhanced Turing Machine models

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3. Scan tapes 1 & 2 simultaneously and delete a single 0 from tape 2 and a single 1 from tape 1.
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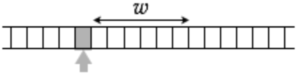
- complexity: $O(n) \quad \Rightarrow \quad L \in \mathrm{TIME}(n^2) \quad \Rightarrow \quad L \in \mathcal{P}$

A pictorial definition of Complexity

Let consider a Turing Machine:



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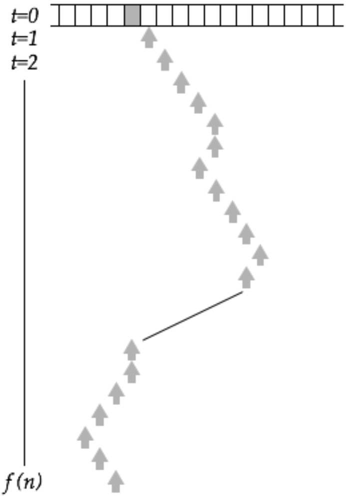

Synthesis and Extension to space complexity

The time-complexity is the number of elementary transitions before reaching a halting state ($f(n)$) where n is the size of the input.

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- Memory is also a critical resource.
- In complexity theory, memory is often referred as *space*.

- How to measure the memory used by an algorithm?

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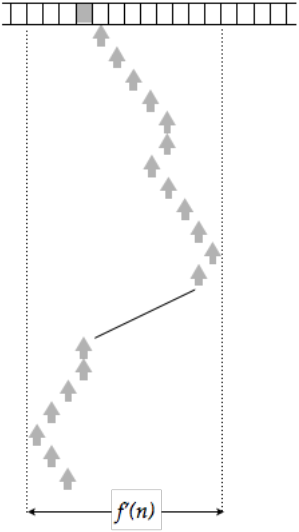
The space-complexity is the number of distinct elementary cells involved before reaching a halting state ($f'(n)$).

A pictorial view of space complexity



PSPACE

Definition

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We define the **space complexity class**

$\mathrm{SPACE}(f(n)) = \{L \mid L$ is a language *decided* by a Turing Machine using $O(f(n))$ cells on a TM, where n is the size of the input$\}$

We are interested in the characterization of the languages –problems– that are decided in polynomial space.

PSPACE is defined similarly to $\mathcal{P}$

PSPACE is the class of *polynomially decidable* languages.

$$PSPACE = \bigcup_k \mathrm{SPACE}(n^k)$$

PolyLog Space classes

The class PSPACE is very large.

We are looking for a restricted class where the number of cells visited during an execution of the TM is bounded by a poly-logarithmic function. We do not count the cells used for coding the input word.

Let us change slightly the definition.

Definition

Let $f : \mathbb{N} \to \mathbb{N}$ be a function.

$\mathrm{SPACE}(f(n)) = \{L \mid L$ is a language *decided* by a 2-tapes TM –where the first tape can not be modified (read only) and the second one is the working tape– using $O(f(n))$ cells on the second tape of this TM, where n is the size of the input$\}$

This does not change the previous definition of PSPACE but it is important for the more refined space classes.

LogSPACE

$LogSPACE$ is defined as $PSPACE(logn)$

We will come back later on this class.

Non-deterministic time complexity class

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We define the **non-deterministic time complexity class**

$\mathrm{NTIME}(f(n)) = \{L \mid L$ is a language *decided* by a non-deterministic Turing Machine in $O(f(n))$ time, where n is the size of the input$\}$

Example:

$\mathrm{HPATH} = \{\langle G, s, t\rangle \mid G$ is a graph with a Hamiltonian path from s to $t\}$

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$M =$ "On input $\langle G, s, t\rangle$:

1. Non-deterministically generate a permutation of the vertex set, $v_1, v_2, \ldots, v_n$.
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- M decides HPATH
- $f(n) = O(n^2) \quad \Rightarrow \quad \mathrm{HPATH} \in \mathrm{NTIME}(n^2)$

Certificates and Verifiers

- "*non-deterministically generate*" a string
- check if the generated string has a certain property of the language
- if this input is in the language, then at least one such string exists
- we call this string a **certificate**

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- Examples of certificates
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- A **polynomial time verifier** runs in polynomial time with respect to the length of the input w

Equivalence of Verifiers and Non-deterministic TM

Theorem

A language L has a polynomial time verifier $\mathcal{V}$ if and only if there is a polynomial time Non-deterministic Turing Machine $NDTM$ which decides it.

Proof: $(\Rightarrow)$ Consider a polynomial time verifier $\mathcal{V}$ for L

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$NDTM =$ "On input w of length n:

1. Non-deterministically generate a string c of length n^k.
2. Run $\mathcal{V}$ on input $\langle w, c \rangle$.
3. If $\mathcal{V}$ accepts, then *accept*, else *reject*."

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$\mathcal{V}$ = "On input $\langle w, c\rangle$:

1. Simulate $NDTM$ on input w using each symbol of c as the non-deterministically choice in order to decide the next step.
2. If this branch of computation accepts, then *accept*, else *reject*."

The class $\mathcal{NP}$

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equivalently

$\mathcal{NP}$ is the class of languages that have a polynomial time verifier.

$\mathcal{P}$ versus $\mathcal{NP}$

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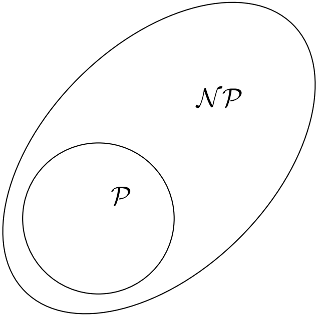
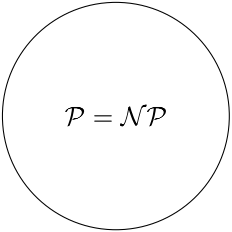


or



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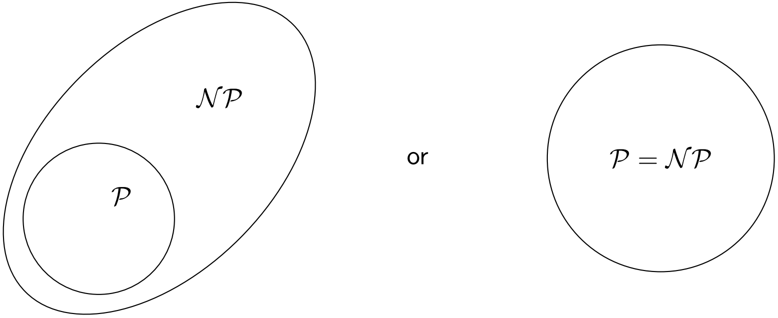

What do we know? $\mathcal{NP} \subseteq \mathrm{EXPTIME} = \bigcup_k \mathrm{TIME}(2^{n^k})$

Coming back to the space complexity: NPSPACE

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Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We define the **space complexity class**

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Some important properties

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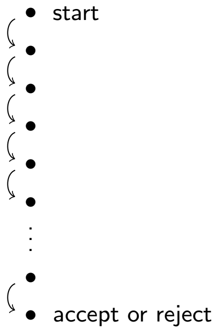
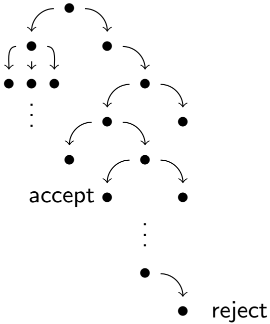
$$NTIME(f(n)) \subset \bigcup_k \mathrm{SPACE}(f(n))$$

Both proofs are intuitive and they are skipped.

Non-deterministic Turing Machines



deterministic computation



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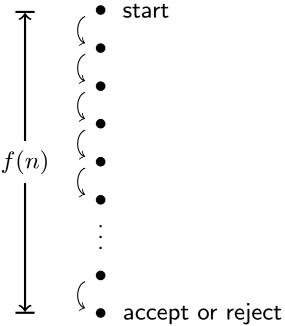

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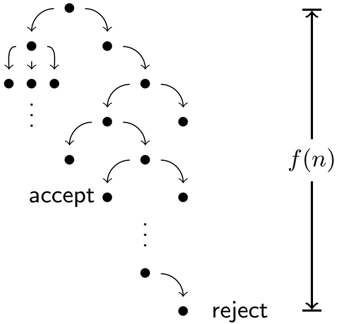

non-deterministic computation

The **running time** of a non-deterministic Turing Machine which *decides* a language is a function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps on any branch of the computation on any input of length n.

non-deterministic space vs time (1)

Let consider again the problem PATH (also called REACH):

- REACH
- Input: a graph G and two vertices s and t
- Question: Is there a path between s and t in G?

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Clearly, REACH $\in \mathcal{P}$

It will play a major role in the class LogSPACE.

non-deterministic space vs time (2)

- The idea is to associate a graph to all possible transitions where ω is the input word.
  This graph has $O(c^{f(n)})$ vertices.
- Then apply REACH on (G_ω, K_ω, H)
  Let us assume WLOG that there is only one acceptance state.
- Accept ω if and only if there exists a path between the original configuration and H

REACH $\in O(n^2)$ in time

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deterministic vs non-deterministic space

- REACH $\in$ SPACE$(log^2(n))$
  The proof is based on a very smart way to progress in the paths.

Savitch Theorem

$$NTIME(f(n)) \subset \bigcup_k \mathrm{SPACE}(f(n))$$

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Corollary

PSPACE = NPSPACE

Synthesis

We have the following series of inclusions:

$$LogSPACE \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq PSPACE = NPSPACE \subseteq EXPTIME$$

Co-$\mathcal{NP}$

Definition

This is the class of languages L whose complements $(\Sigma^* - L)$ are in $\mathcal{NP}$

Let consider the following problem:

- PRIME
- **Instance:** an integer N
- **question:** Is N a prime?

Coding the input

- There exists a well-known algorithm for solving this problem
- The Erathostenes' sieve
  whose running time is in $O(\sqrt{N})$

Is the problem clearly in $\mathcal{P}$?

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Is the problem clearly in $\mathcal{P}$? **Not clearly!**

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This means that the complexity of this algorithm is exponential!

Coding the input

- There exists a well-known algorithm for solving this problem
- The Erathostenes' sieve
  whose running time is in $O(\sqrt{N})$

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$\textsc{Prime}$ is an example of a problem that is hard to give a certifier...
Hard to prove that it is in $\mathcal{NP}$ (but its complement –recognize that an integer is NOT a prime– is very simple by multiplying its divisers).

Co-$\mathcal{NP}$

Let us remark that if a problem is in $\mathcal{NP}$, its complement is not necessarily in $\mathcal{NP}$...

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For deterministic TM, we have $\mathcal{P} =$ co-$\mathcal{P}$

An open question

$$\mathcal{P} = ?\ \mathcal{NP} \bigcap \text{co-}\mathcal{NP}$$