Fundamental Computer Science
Turing Machines
Training session

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MoSIG1-M1Info, 2021

Simple exercises

Aim.

Manipulate MT.

Construct the Turing Machines that implement the following operations[1].

1. copy reversed
   from $\sqcup w \underline{\sqcup}$ to $\sqcup w w^{rev} \underline{\sqcup}$
2. right shift
   from $\sqcup w \underline{\sqcup}$ to $\sqcup \sqcup w \underline{\sqcup}$
3. delete w
   from $\sqcup w \underline{\sqcup}$ to $\sqcup \underline{\sqcup}$

[1] The solution is left to the readers (easy).

Exercise:

Aim.

Strenghten the formalism.

Give the high-level description for a Turing Machine that accepts the following language

$L = \{\#x_1\#x_2\#\ldots\#x_\ell :$ each $x_i \in \{0,1\}^*$ and $x_i \neq x_j$ for each $i \neq j\}$

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Analysis

- All pairs (x_i, x_j) must be compared.
- For each pair, the bits must be tested one by one.

Exercise

Consider the Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$
where $K = \{q_0, q_1, q_2, h\}$,
$\Sigma = \{a\}$,
$\Gamma = \{a, \sqcup, \#\}$,
$s = q_0$ and $H = \{h\}$
δ is given by the following table.

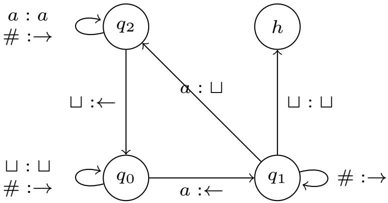
q	q_0	q_0	q_0	q_1	q_1	q_1	q_2	q_2	q_2
σ	a	$\sqcup$	$\#$	a	$\sqcup$	$\#$	a	$\sqcup$	$\#$
$\delta(q, \sigma)$	$(q_1, \leftarrow)$	$(q_0, \sqcup)$	$(q_0, \rightarrow)$	$(q_2, \sqcup)$	$(h, \sqcup)$	$(q_1, \rightarrow)$	(q_2, a)	$(q_0, \leftarrow)$	$(q_2, \rightarrow)$

Let $n \geq 0$.
Describe what M does when started in the configuration $(q_0, \#a^n\underline{a})$.

Solution

$$\Sigma = \{a\}, \quad \Gamma = \{a, \#, \sqcup\}, \quad s = q_0, \quad H = \{h\}$$

Let draw the state graph.



$$(q_0, \#a^n\underline{a})$$

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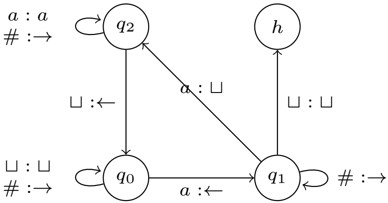

$$
\begin{aligned}
(q_0, \#a^n\underline{a}) &\vdash_M (q_1, \#a^{n-1}\underline{a}a) \\
&\vdash_M (q_2, \#a^{n-1}\underline{\sqcup}a) \\
&\vdash_M (q_0, \#a^{n-2}\underline{a} \sqcup a) \\
&\vdash_M (q_1, \#a^{n-3}\underline{a}a \sqcup a) \\
&\vdash_M (q_2, \#a^{n-3}\underline{\sqcup}a \sqcup a) \\
&\vdash_M (q_0, \#a^{n-4}\underline{a} \sqcup a \sqcup a) \\
&\vdash_M (q_1, \#a^{n-5}\underline{a}a \sqcup a \sqcup a) \\
&\vdash_M (q_2, \#a^{n-5}\underline{\sqcup}a \sqcup a \sqcup a) \\
&\ldots
\end{aligned}
$$

More exercises

1. Give the full details of the following three Turing Machines.

$$> LL \qquad\qquad > \overset{\circlearrowright}{R} \qquad\qquad > L \xrightarrow{\sqcup} R$$

2. Explain what the following Turing Machine does.

$$> R \xrightarrow{a \neq \sqcup} > R \xrightarrow{b \neq \sqcup} > R_{\sqcup} a R_{\sqcup} b$$

Finding the MAX

Give the high-level definition of a Turing Machine that finds the maximum between three integers encoded in *unary*.

What is the length of the computation?

Exercise

Prove that the language $L = \{a^n b^n : n \geq 0\}$ is decidable.

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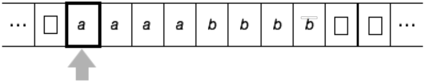
Let us study an example.



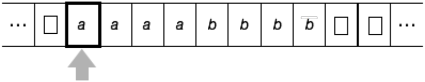
Solution

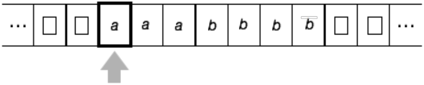
A solution is to decompose the operations: establish successive one-to-one correspondences between each pair of a and b

States

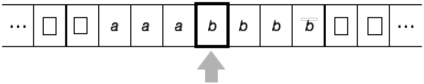
- go through the word $aa \cdots a$ until its end (from left to right), resp. with $bb \cdots b$
- similar operations backwards on both words
- q_0 denotes the initial state
- q_R is the rejected state and q_{acc} the acceptance state

Detailed moves

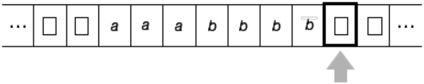




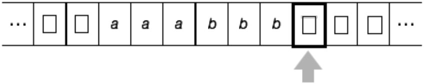
…
a
a
a
b
b
b
b
…



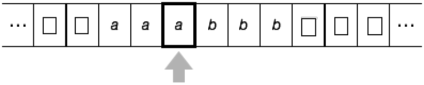
…
□
□
a
a
a
b
b
b
b
□
□
…

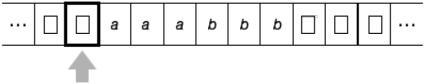


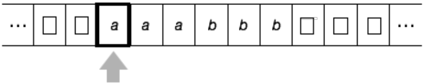
a a a b b b b



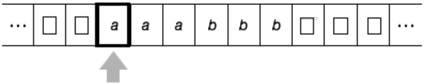
a
a
a
b
b
b


a a a b b b


a a a b b b



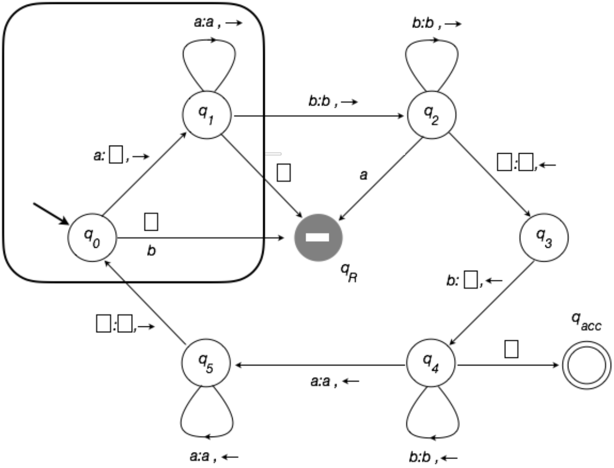
and so on...



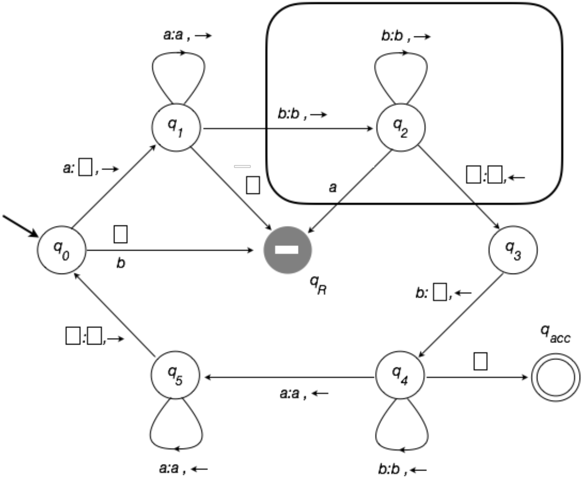
and so on...

Let us draw the Turing Machine.

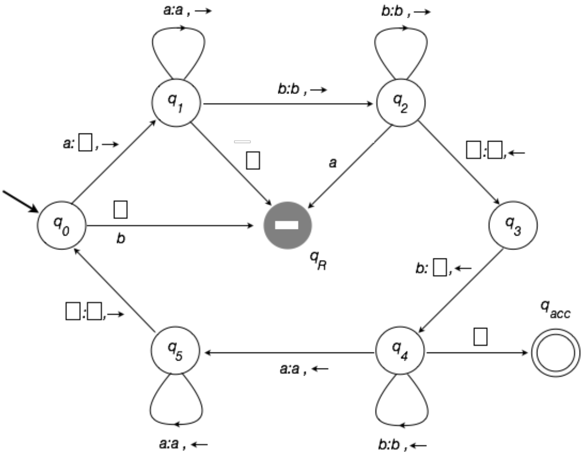
Go through the word $aa \cdots a$



Go through the word $bb \cdots b$



The complete picture



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Solution: We just need to give a Turing Machine that decides it. (give a Turing Machine composed by simple Turing Machines as described previously)

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