Fundamental Computer Science
Sequence 1: Turing Machines
Classical extensions

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Agenda

Objective of the session

Study the most common extensions of **Turing machines**.

See Pierre Wolper, Introduction à la calculabilité or any related book.

Extensions of the Turing Machine

We have already presented an extension:

- write in the tape and move left or right at the same time
- modify the definition of the transition function

  initial: from $(K \setminus H) \times \Gamma$ to $K \times (\Gamma \cup \{\leftarrow, \rightarrow\})$

  extended: from $(K \setminus H) \times \Gamma$ to $K \times \Gamma \times \{\leftarrow, \rightarrow\}$

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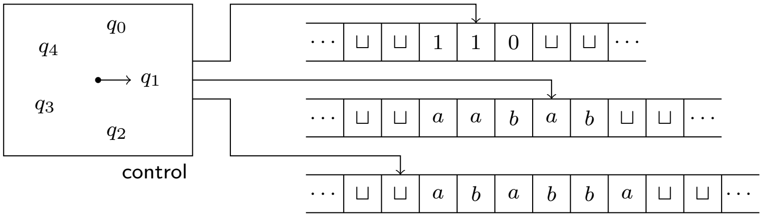
  extended: from $(K \setminus H) \times \Gamma$ to $K \times \Gamma \times \{\leftarrow, \rightarrow\}$

- if the extended Turing Machine halts on input w after t steps, then the initial Turing Machine halts on input w after at most $2t$ steps

Multiple tapes

A k-tape Turing Machine (M) is a sextuple $(K, \Sigma, \Gamma, \delta, s, H)$, where K, Σ, Γ, s and H are as in the definition of the ordinary Turing Machine, and δ is a transition function

$$\text{from} \quad (K \setminus H) \times \Gamma^k \quad \text{to} \quad K \times (\Gamma \cup \{\leftarrow, \rightarrow\})^k$$



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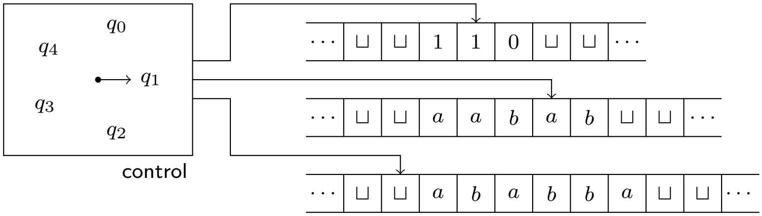


Multiple tapes

Theorem

Every k-tape, $k > 1$, Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$ has an equivalent single tape Turing Machine $M' = (K', \Sigma', \Gamma', \delta', s', H')$.

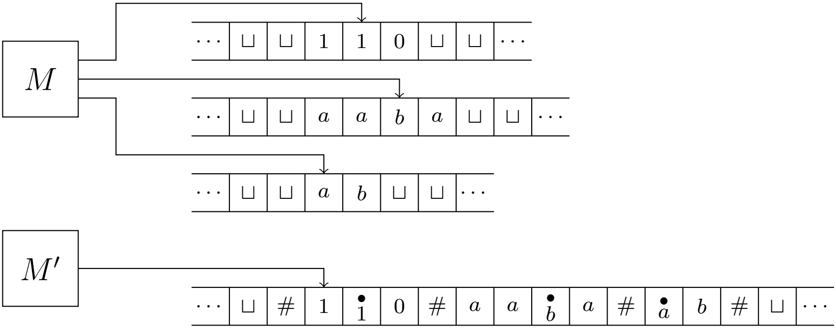
If M halts on input $w \in \Sigma^*$ after t steps, then M' halts on input w after $O(t(|w| + t))$ steps.

Sketch of the proof:

- M' simulates M in a single tape
- $\#$ is used as delimiter to separate the contents of different tapes
- dotted symbols are used to indicate the actual position of the head of each tape
  - for each symbol $\sigma \in \Gamma$, add both σ and $\dot{\sigma}$ in Γ'
- use the same set of halting states

Multiple tapes

Sketch of the proof:



Multiple tapes

Sketch of the proof:

$M' =$ "On input $w = w_1 w_2 \dots w_n$:

1. Format the tape to represent the k tapes:

$$\# \dot{w}_1 w_2 \dots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \dots \#$$

2. For each step that M performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of M.

Multiple tapes

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3. If at any point there is a need to move a virtual head outside the area marked for the corresponding tape, then shift right the contents of all tapes succeeding."

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What is the number of steps for M'?

1. $O(|w|)$

2. & 3. $O(|w| + t)$ per step $\Rightarrow O(t(|w| + t))$ in total
   - size of the tape no more than $O(|w| + t)$

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... and it can be used to simulate functions in an easier way
(a function can use one or more not used tapes)

Multiple tapes: example with $k = 2$ tapes

$$
\begin{tikzcd}
> R^{1,2} \arrow[r, "\sigma^1 \neq \sqcup"] \arrow[rr, bend right, "\sqcup^1"'] & \sigma^2 \arrow[l, bend right] & L^2_{\sqcup}\; R^{1,2} \arrow[r, "\sigma^2 \neq \sqcup"] & \sigma^1 \arrow[l, bend right]
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- extend notation:
  - $R^{1,2}$: move the head of both tapes to the right
  - σ^2 (as a state): write the symbol σ in tape 2
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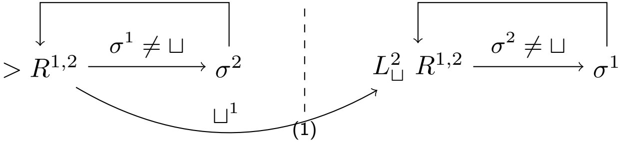

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after (1)		

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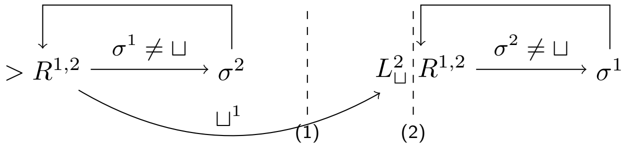

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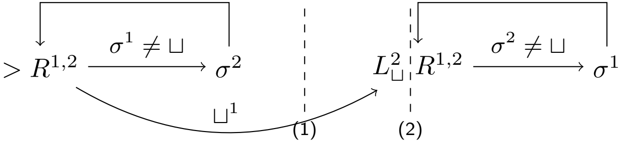

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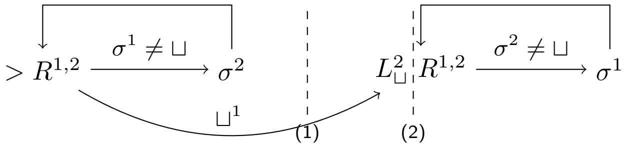

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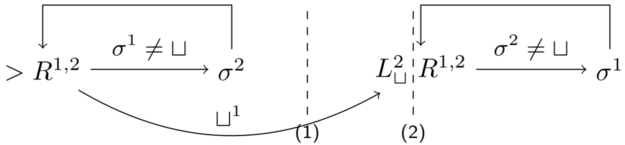

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transforms w to $w \sqcup w$

Another extension: Multiple heads

Definition (informal)

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- we need a convention if two heads try writing at the same place

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Theorem

Every multiple head Turing Machine M has an equivalent single head Turing Machine M'.

The simulation by M' of M on an input w which leads to a halting state takes time quadratic to the size of the input $|w|$ and the number of steps t that M performs.

Proof (sketch):

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$\cdots$	$\sqcup$	m	y	$\sqcup$	i	n	p	u	t	$\sqcup$	$\cdots$
			$\wedge$								
							$\wedge$				
				$\wedge$							

Multiple heads: example

Give a Machine Turing with two heads that transforms the input $\underline{\sqcup} w$ to $\overline{\underline{\sqcup}} w \sqcup w$.

- extend notation:
  - $\underline{\sigma}$, $\overline{\sigma}$, $\overline{\underline{\sigma}}$: the position of the 1st, 2nd and both heads, respectively
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Unbounded tapes

What happens if the tape is bounded in one direction?

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Theorem

Every two-direction unbounded tape Turing Machine M has an equivalent single-direction unbounded tape Turing Machine.

Two-dimensional tape

Definition (informal)

- move the head left/right/up/down

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Theorem

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The simulation by M' of M on an input w which leads to a halting state takes time polynomial to the size of the input $|w|$ and the number of steps t that M performs.

Proof (sketch):

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Theorem

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The simulation by M' of M on an input w which leads to a halting state takes time polynomial to the size of the input $|w|$ and the number of steps t that M performs.

Proof (sketch):

- use a multiple tape Turing Machine
- each tape corresponds to one line of the two-dimensional memory

Discussion

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- Observation: a computation in the prototype Turing Machine needs a number of steps which is bounded by a polynomial of the size of the input and of the number steps in any of the extended model