JOSEPHUS PROBLEM AND MASTER THEOREM

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1 Josephus' problem

The problem comes from an old story reported by Flavius Josephus during the Jewish-Roman war in the first century. The legend reports that Flavius was among a band of 41 rebels trapped in a cave by the roman army. Preferring suicide to capture, the rebels decided to form a circle and proceeding around to kill every second remaining person until no one was left. As Josephus had some skills in Maths and wanted none of this suicide non-sense, he quickly calculated where he should stand in the circle in order to stay alive at the end of the process.

Definition. Given n successive numbers in a circle. The problem is to determine the *survival number* (denoted by $J(n)$) in the process of removing every second remaining number starting from 1 (see figure 1).

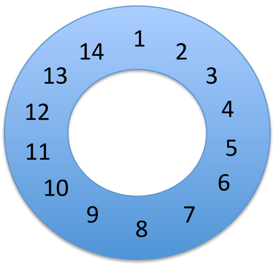

Figure 1: Initial situation for the Josephus process.

In particular, we are going to determine if there exists a closed formula. Guessing the answer sounds not obvious. We need to better understand the progression.

Property 1. $J(n)$ is odd

We called *round*, the set of steps to come back at a given position in the circle. Starting at 1, the first round is completed after $\lceil \frac{n}{2} \rceil$ steps. Then, again half of the of the remaining numbers are removed in the second round and so on.

How many rounds do we have for determining $J(n)$?

Property 2. (even numbers) Show $J(2n) = 2J(n) - 1$

Deduce $J(2^m) = 1$ for all m.
Let us turn to odd numbers.

Property 3. (odd numbers) $J(2n+1) = 2J(n) + 1$

Show **Property 4.** $J(2^m + k) = 2k + 1$

We can even go one step further with this problem by remarking that powers of 2 play an important role. Let us use the radix 2 representation of n and $J(n)$:
$n = \sum_{j=0}^{j=m} b_j.2^j = b_m.2^m + b_{m-1}.2^{m-1} + ... + b_1.2 + b_0$

2 Master Theorem

The cost analysis of the divide-and-conquer paradigm leads to the following recurrence equation. We assume that n is a power of b (in other words, it can be perfectly divided by b until reaching the value 1. $a \geq 1$ et $b > 1$.

- $F(n) = \Theta(1)$ if $n \leq n_0$
- $F(n) = a.F(\frac{n}{b}) + f(n)$

We give below the general formulation for solving this equation:

1. if $f(n) \in \mathcal{O}(n^{log_b a - \epsilon})$ then $F(n) \in \Theta(n^{log_b a})$
2. if $f(n) \in \Theta(n^{log_b a})$ then $F(n) \in \Theta(n^{log_b a} log(n))$
3. if $f(n) \in \Omega(n^{log_b a + \epsilon})$ and if $a.f(n/b) \leq cf(n)$ for some constant $c < 1$ then, $F(n) \in \Theta(f(n))$

where ϵ is a positive real number.

2.1 Analysis of regular (well-balanced) cases

- $F(1) = 1$
- $F(n) = a.F(\frac{n}{b}) + c$ (c is a positive constant)

Solve this problem for any (a, b).

- $F(1) = 1$
- $F(n) = a.F(\frac{n}{b}) + n$

Solve this problem for $a = b = 2$

- $F(1) = 1$
- $F(n) = a.F(\frac{n}{b}) + f(n)$

Solve this problem for a (positive) multiplicative function f defined on the successive powers of b.

Give an interpretation of the Master Theorem.