Lecture 1 – Maths for Computer Science

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Lecture notes MoSIG1

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Content

Sum of two consecutive triangular numbers

The problem. Compute $\Delta_n + \Delta_{n-1}$

Computing the first ranks leads us to an evidence: $\Delta_1 + \Delta_0 = 1$, then, $3 + 1 = 4$, $6 + 3 = 9$, $10 + 6 = 16$, 25, 36, ...

It is *natural* to guess $\Delta_n = n^2$, which is easy provable by induction (or alternatively, directly using the expression $\Delta_n + \Delta_{n-1} = n + 2\Delta_{n-1}$ since $\Delta_n = n + \Delta_{n-1}$).

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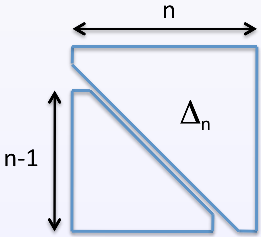
$\Delta_n + \Delta_{n-1} = n + 2\Delta_{n-1}$ since $\Delta_n = n + \Delta_{n-1}$).
This result can be directly obtained using a geometric pattern:



Content

Content

Four consecutive numbers

Let denote $P_n = n.(n+1).(n+2).(n+3)$

The problem:

Study some properties of P_n

In particular, the two following properties:

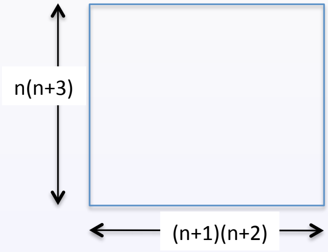
1. P_n is equal to a square minus 1
2. $P_n = n.(n+1).(n+2).(n+3)$ is divisible by 4!

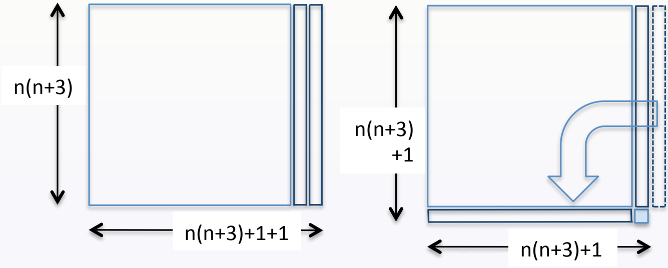
$P(n) + 1$ is a perfect square.

Let check at the first ranks:
$P_2 = 2 \times 3 \times 4 \times 5 = 120 = 15 \times 8 = 11^2 - 1$
$P_3 = 3 \times 4 \times 5 \times 6 = 360 = 45 \times 8 = 19^2 - 1$
$P_4 = 4 \times 5 \times 6 \times 7 = 840 = 105 \times 8 = 29^2 - 1$
...

We propose a graphical proof.

Compare the extreme product $n(n+3)$ to the medium one $(n+1)(n+2)$:
$n(n+3) = n^2 + 3n$ and $(n+1)(n+2) = n^2 + 3n + 2$.





Thus, $P(n) = ((n.(n+3)+1)^2 - 1$

$P_n = n.(n+1).(n+2).(n+3)$ is divisible by 4!

$P(n)$ is divisible by 3 since there is at least one multiple of 3 in three (thus, four) consecutive products.
We prove that it is also divisible by 8:

In the product $P(n)$, there are exactly 2 even numbers:
$2k$ and $2k+2$, thus their product is equal to:
$2k.2(k+1) = 4.k.(k+1) = 8.\Delta_k$

As 3 and 8 have no common divisors, $P(n)$ is divisible by their product 24.

Content

Square of odd numbers

An intermediate result.

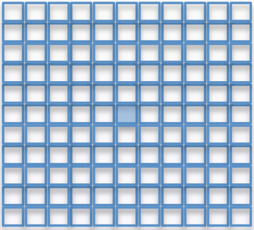
The problem:

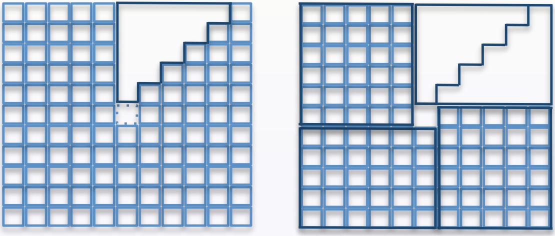
Show that the squares of odd numbers are multiples of 8 minus 1.

Using a geometrical argument.

This is the case for $3^2 = 8 + 1$, $5^2 = 3 \times 8 + 1$, $7^2 = 6 \times 8 + 1$, etc.

The result can also easily be proven analytically as follows:
$(2q + 1)^2 = 4q^2 + 4q + 1 = 4q(q + 1) + 1$.
As one of the consecutive numbers q and $q + 1$ is even, we obtain:
$(2q + 1)^2 = 8k + 1$.





At this point, we can establish a deeper link with triangular numbers.
Looking more carefully at the previous figure, we remark that each of the 8 quadrants is composed of a series of 1 unit square, followed by 2 unit squares and so on, up to q unit squares.

This can be easily proved by the relation:
$\frac{(2q+1)^2-1}{8} = \frac{4q^2+4q+1-1}{8} = \frac{4q(q+1)}{8} = \Delta_q$.

Content

Tetrahedral numbers

The problem:

The sum of the Δ_n is denoted by Θ_n: $\Theta_n = \sum_{k=1}^{n} \Delta_k$.

A way to compute it is to consider 3 copies of Θ_n and organize them smartly[1] as a triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 2 & & 3 & & \\
 & 1 & & 2 & & 3 & & \dots & \\
1 & & 2 & & 3 & & \dots & & n
\end{array}
$$

[1] the way we established the closed formula for triangular numbers was based on 2 copies arranged up side down

The proof is obtained by Fubini's principle by rotating this triangle as shows below:

1	1	n
1 2	2 1	n-1 n-1
1 2 3	3 2 1	... ...
1 2 3 ...	... 3 2 1	2 2 2 2
1 2 3 ... n	n ... 3 2 1	1 1 1 1 1

The first row is equal to $n + 2$.
The second one is equal to $3 + 2(n - 1) + 3 = 2(n + 2)$.

The proof is obtained by Fubini's principle by rotating this triangle as shows below:

1	1	n
1 2	2 1	n-1 n-1
1 2 3	3 2 1	... ...
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1 2 3 ... n	n ... 3 2 1	1 1 1 1 1

The first row is equal to $n+2$.
The second one is equal to $3+2(n-1)+3 = 2(n+2)$.
Let us sum up the elements in row k:
$\Delta_k + k(n-k+1) + \Delta_k = k(k+1) + kn - k^2 + k = k(n+2)$.
Thus, the global sum is equal to $n+2$ times $(1+2+...+n)$.
Finally, $3\Theta_n = (n+2)\Delta_n$

Synthesis

We proved some results in this chapter, in particular:

- $Id_n = 1 + 1 + ... + 1 = n$
- $\Delta_n = 1 + 2 + 3 + ... + n = \frac{1}{2}.Id_n.(n+1)$
- $\Theta_n = \Delta_1 + \Delta_2 + ... + \Delta_n = \frac{1}{3}.\Delta_n.(n+2)$

A natural question is if we can go further following the same pattern for computing $\sum_{k=1}^{n} \Theta_k$, and so on. The next ones are the *pentatope* numbers (defined by Π_n), defined as the sum of Θ_n.

More properties

If we write these numbers as polynomials of n, we obtain:

- Rank 1. $Id_n = n$
- Rank 2. $\Delta_n = \frac{1}{2}n(n+1)$
- Rank 3. $\Theta_n = \frac{1}{6}n(n+1)(n+2)$ where $6 = 1 \times 2 \times 3$.
- Rank 4. $\Pi_n = \frac{1}{24}n(n+1)(n+2)(n+3)$ where $24 = 1 \times 2 \times 3 \times 4$.
- The next one (rank 5) is $\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4)$

As these numbers are integers, $P(n) = n(n+1)(n+2)(n+3)$ is a multiple of 4!

Content

Sum of squares

Link with triangular and tetrahedral numbers

$\square_n = \Delta_n + (\Delta_n - \Delta_1) + (\Delta_n - \Delta_2) + ... + (\Delta_n - \Delta_{n-1})$

$= n.\Delta_n - \sum_{1 \leq k \leq n-1} \Delta_k$

$= n.\Delta_n - \Theta_{n-1}$

$= n\frac{n(n+1)}{2} - \frac{(n-1)n(n+1)}{6}$

$= \frac{n(n+1)}{6}(3n - (n-1))$

$= \frac{n(n+1)}{6}(2n+1)$

$= \frac{n(n+\frac{1}{2})(n+1)}{3}$

A nice property of pyramid numbers

The problem. Compute the sum of two consecutive tetrahedral numbers: $\Theta_n + \Theta_{n-1}$ (similarly as what we did for $\Delta_n + \Delta_{n-1}$).

Recall the first pyramid numbers: $1, 5, 14, 30, 55, \ldots$.

It is equal to $\square_n$

The proof is straightforward by applying the definition:
$\Theta_n = \sum_{k=1}^{n} \Delta_k$

$\Theta_n + \Theta_{n-1}$
$= (\Delta_n + \Delta_{n-1}) + (\Delta_{n-1} + \Delta_{n-2}) + \ldots + (\Delta_2 + \Delta_1) + \Delta_1$
$= n^2 + (n-1)^2 + \ldots + 2^2 + 1 = \square_n$

Is there any equivalent of the sum of odds?

We established a strong link between sum of odds and the sum of consecutive triangular numbers.

According to the binomial expression, the next step is to compute the sum of hexagonal numbers $3n^2 + 3n + 1$.
$(n+1)^2 = n^2 + 2n + 1$
$(n+1)^3 = n^3 + 3n^2 + 3n + 1$

Do you think that such numbers are linked with the sum of two consecutive tetrahedral numbers?