Lecture 1 – Maths for Computer Science
Computing the sum of cubes

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The problem

Definition:
Sum of the n first cubes:
$C_n = \sum_{k=1}^{n} k^3$.

Determine the asymptotic behavior

A rather simple analysis of lower and upper bounds leads to:
$C_n = \Theta(\frac{n^4}{4})$

A first brute force method

Similarly to the problem of computing the sum of squares, we may use the method of undetermined coefficients:

Let write $C_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3 + +\alpha_4 n^4$

This method requires to solve a 5 by 5 system of linear equations (that can be simplified into a 4 by 4 system since $\alpha_0 = 0$).

Let us take the time to observe the sum more carefully...

Computing C_n on the first ranks

$$
\begin{array}{lcrcr}
C_1 & = & 1 & & \\
C_2 & = & 1+8 & = & 9 \\
C_3 & = & 9+27 & = & 36 \\
C_4 & = & 36+64 & = & 100 \\
C_5 & = & 100+125 & = & 225 \\
\vdots & & \vdots & &
\end{array} \tag{1}
$$

We remark that the C_k are perfect squares and more precisely, the squares of Δ_k.

The simplified problem

Proposition:

For all k,

$$\Delta_k^2 \;=\; \sum_{i=1}^{\Delta_k} (2i-1) \;=\; \sum_{i=1}^{k} i^3 \qquad (2)$$

Proof:

The left-hand equation in (2) follows from the proposition that gives the sum of the first odd integers (here up Δ_k), thus, they sum to Δ_k^2.

We focus on the right-hand equation.

We take the odd integers in order and arrange them into groups whose successive sizes increase by 1 at each step, as follows:

$$\begin{array}{lrrrr} \text{group 1 (size 1):} & 1, & & & \\ \text{group 2 (size 2):} & 3, & 5, & & \\ \text{group 3 (size 3):} & 7, & 9, & 11, & \\ \text{group 4 (size 4):} & 13, & 15, & 17, & 19 \\ \vdots & \vdots & & & \end{array} \tag{3}$$

Features of Table (3)

We observe first that[1] the i elements of the ith group add up to i^3:

$$\begin{array}{lrrrrl}
\text{group 1 (size 1):} & 1, & & & & : \text{sum} = 1^3 \\
\text{group 2 (size 2):} & 3, & 5, & & & : \text{sum} = 2^3 \\
\text{group 3 (size 3):} & 7, & 9, & 11, & & : \text{sum} = 3^3 \\
\text{group 4 (size 4):} & 13, & 15, & 17, & 19 & : \text{sum} = 4^3
\end{array}$$

[1] at least within the illustrated portion of the table

We observe next that, by construction, the ith group/row of odd integers in the table consists of the i consecutive odd numbers beginning with the $(\Delta_{i-1}+1)$th odd number, namely, $2\Delta_{i-1}+1$.

Since consecutive odd numbers differ by 2, this means that the ith group (for $i > 1$) comprises the following i odd integers:

$$2\Delta_{i-1}+1,\ 2\Delta_{i-1}+3,\ 2\Delta_{i-1}+5,\ \ldots,\ 2\Delta_{i-1}+(2i-1)$$

Therefore, the *sum* of the i integers in group i, call it σ_i, equals

$$\begin{aligned} \sigma_i &= 2i\Delta_{i-1} + \big(1+3+\cdots+(2i-1)\big) \\ &= 2i\Delta_{i-1} + (\text{the sum of the first } i \text{ odd numbers}) \\ &= 2i\Delta_{i-1} + i^2 \end{aligned}$$

By direct calculation, then,

$$\sigma_i = 2i \cdot \frac{i(i-1)}{2} + i^2 = (i^3 - i^2) + i^2 = i^3$$

The proof is now completed by concatenating the rows of Table (3) and observing the pattern that emerges:

$$(1) + (3+5) + (7+9+11) + \cdots = 1^3 + 2^3 + 3^3 + \cdots$$

$\square$

We now present a *pictorial* proof, namely, the relation between sums of perfect cubes and squares of triangular numbers.

This illustration provides a *non-textual* way to understand this result, and it provides a fertile setting for seeking other facts of this type.
For all k,

$$1^3 \ + \ 2^3 \ + \cdots + \ k^3 \ \ = \ \ \Delta_k^2$$

Proof

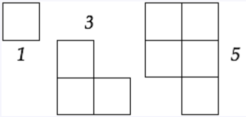
We develop an induction that reflects the structure of Table (3).

Base case.

$$1^3 = 1 = \Delta_1^2$$

While this first (and obvious) case is enough for the induction, it does not tell us much about the structure of the problem.
Therefore, we consider also the next step $k = 2$:

$$1^3 + 2^3 = 9 = \Delta_2^2$$



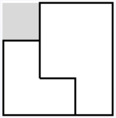


Figure: (Left) set $\{1\}$ of group 1 and the set $\{3, 5\}$ of group 2. (Right) how to form a 3×3 square by pictorially summing the numbers 1, 3, and 5.

Inductive hypothesis. Assume that the target equality holds for all $i < k$; i.e.,

$$1^3 \; + \; 2^3 \; + \cdots + \; i^3 \;\; = \;\; \Delta_i^2$$

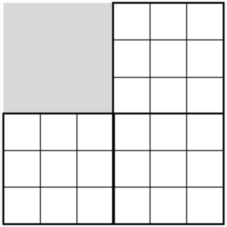
If we go one step further, to incorporate group 3, i.e., the set $\{7, 9, 11\}$, into our pictorial summation process, then we discover that mimicking the previous process is a bit more complicated here. More complicated manipulation required to form the $\Delta_3 \times \Delta_3$ square is a consequence of the odd cardinality of the group-3 set. We must extend our induction for the cases of odd and even k.

Inductive extension for odd k.

$$\Delta_k^2 \;\; = \;\; \Delta_{k-1}^2 \; + \; k^3$$

We begin to garner intuition for this extension by comparing the quantities Δ_k^2 and $1 + 2^3 + \cdots + k^3$.

Moving to the pictorial domain, we write k^3 as $k \times k^2$, and we distribute $k \times k$ square blocks around the $\Delta_{k-1} \times \Delta_{k-1}$ square, as shown below for the case $k = 3$.

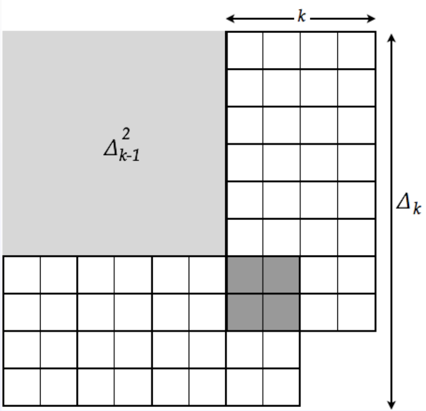


Because k is odd, the small squares pack perfectly since $(k-1)$ is even, hence divisible by 2.
The depicted case depicts pictorially the definition of triangular numbers: $k \cdot \frac{1}{2}(k-1) \ = \ \Delta_{k-1}$.

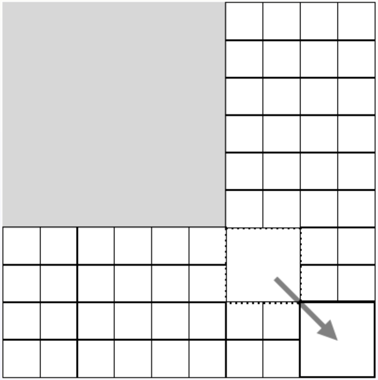
Inductive extension for even k.

The basic reasoning here mirrors that for odd k, with one small difference.

Now, as we assemble small squares around the large square, two subsquares overlap, as depicted below.



We must manipulate the overlapped region in order to get a tight packing around the large square.



Happily, when there is a small overlapping square region, there is also an identically shaped empty square region, as suggested by these two figures.

More details.

Because $(k-2)$ is even, the like-configured square blocks can be allocated to two sides of the initial $\Delta_{k-1} \times \Delta_{k-1}$ square (namely, its right side and its bottom).

The overlap has the shape of a square that measures $\frac{1}{2}(\Delta_k - \Delta_{k-1})$ on a side.

One also sees in the figure an empty square in the extreme bottom right of the composite $\Delta_k \times \Delta_k$ square, which matches the overlapped square identically. This situation is the pictorial version of the equation

$$\Delta_k^2 \; - \; \Delta_{k-1}^2 \;\; = \;\; \frac{1}{4}k^2\left((k+1)^2 - (k-1)^2\right) \;\; = \;\; k^3$$

We have thus extended the inductive hypothesis for both odd and even k, whence the result. $\square$