Training Lecture 1 – Maths for Computer Science Correction

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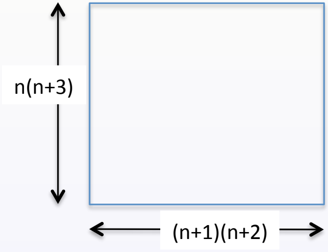
sept. 2018

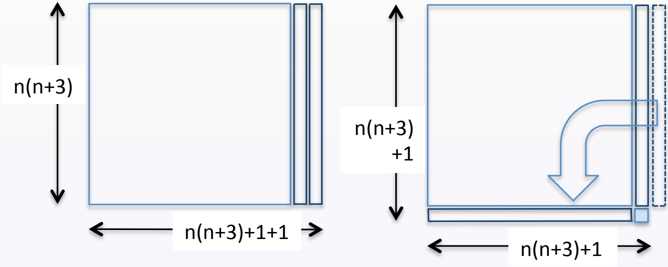
Content

Four consecutive numbers

First, we remark that $P_n = n.(n+1).(n+2).(n+3)$ is divisible by 3 since there is at least one multiple of 3 in the four consecutive products. Then, we prove that it is also divisible by 8. It is easy by decomposing it into two properties:

P_n is equal to a square minus 1. For instance:
$P_3 = 3 \times 4 \times 5 \times 6 = 19^2 - 1$ or $P_7 = 7 \times 8 \times 9 \times 10 = 71^2 - 1$.
Let us compare the extreme product $n(n+3)$ to the medium one $(n+1)(n+2)$:
$n(n+3) = n^2 + 3n$ and $(n+1)(n+2) = n^2 + 3n + 2$.


n(n+3)
(n+1)(n+2)


n(n+3)
n(n+3)+1+1
n(n+3)
+1
n(n+3)+1

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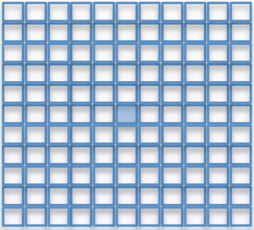
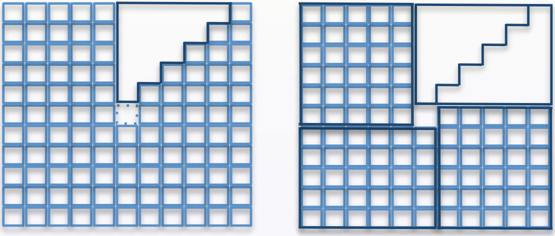
Square of odd numbers

Now, we show by a geometrical proof that the squares of odd numbers are congruent to 1 modulo 8. This is the case for $3^2 = 8 + 1$, $5^2 = 3 \times 8 + 1$, $7^2 = 6 \times 8 + 1$, etc.[1]

[1] The result can also be proved analytically as follows: $(2q+1)^2 = 4q^2 + 4q + 1 = 4q(q+1) + 1$. As one of the consecutive numbers q and $q+1$ is even, we obtain: $(2q+1)^2 = 8k + 1$.





At this point, we can establish a deeper link with triangular numbers.
Looking more carefully at the previous figure, we remark that each of the 8 quadrants is composed of a series of 1 unit square, followed by 2 unit squares and so on, up to q unit squares.

This can be easily proved by the relation:
$\frac{(2q+1)^2-1}{8} = \frac{4q^2+4q+1-1}{8} = \frac{4q(q+1)}{8} = \Delta_q$.

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Tetrahedral numbers

Definition:
The sum of the Δ_n is denoted by Θ_n: $\Theta_n = \sum_{k=1}^{n} \Delta_k$.

A way to compute it is to consider 3 copies of Θ_n and organize them smartly[2] as a triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 2 & & 3 & & \\
 & 1 & & 2 & & 3 & & \dots & \\
1 & & 2 & & 3 & & \dots & & n
\end{array}$$

[2] the way we established the closed formula for triangular numbers was based on 2 copies arranged up side down

The proof is obtained by Fubini's principle by rotating this triangle as shows below:

1	1	n
1 2	2 1	n-1 n-1
1 2 3	3 2 1	... ...
1 2 3 ...	... 3 2 1	2 2 2 2
1 2 3 ... n	n ... 3 2 1	1 1 1 1 1

The first row is equal to $n + 2$.
The second one is equal to $3 + 2(n-1) + 3 = 2(n+2)$.

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1 2 3 ...	... 3 2 1	2 2 2 2
1 2 3 ... n	n ... 3 2 1	1 1 1 1 1

The first row is equal to $n+2$.
The second one is equal to $3+2(n-1)+3=2(n+2)$.
Let us sum up the elements in row k:
$\Delta_k + k(n-k+1) + \Delta_k = k(k+1) + kn - k^2 + k = k(n+2)$.
Thus, the global sum is equal to $n+2$ times $(1+2+...+n)$.
Finally, $3\Theta_n = (n+2)\Delta_n$

Synthesis

We proved some results in this chapter, in particular:

- $Id_n = 1 + 1 + ... + 1 = n$
- $\Delta_n = 1 + 2 + 3 + ... + n = \frac{1}{2}.Id_n.(n+1)$
- $\Theta_n = \Delta_1 + \Delta_2 + ... + \Delta_n = \frac{1}{3}.\Delta_n.(n+2)$

A natural question is if we can go further following the same pattern for computing $\sum_{k=1}^{n} \Theta_k$, and so on. The next ones are the *pentatope* numbers (defined by Π_n), defined as the sum of Θ_n.

More properties

If we write these numbers as polynomials of n, we obtain:

- Rank 1. $Id_n = n$
- Rank 2. $\Delta_n = \frac{1}{2}n(n+1)$
- Rank 3. $\Theta_n = \frac{1}{6}n(n+1)(n+2)$ where $6 = 1 \times 2 \times 3$.
- Rank 4. $\Pi_n = \frac{1}{24}n(n+1)(n+2)(n+3)$ where $24 = 1 \times 2 \times 3 \times 4$.
- The next one (rank 5) is $\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4)$

As these numbers are integers, $n(n+1)(n+2)(n+3)$ is a multiple of $24 = 1 \times 2 \times 3 \times 4$.
Let us denoted this number by P_n.

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Sum of squares

Link with triangular and tetrahedral numbers

$$
\begin{aligned}
\square_n &= \Delta_n + (\Delta_n - \Delta_1) + (\Delta_n - \Delta_2) + \ldots + (\Delta_n - \Delta_{n-1}) \\
&= n.\Delta_n - \textstyle\sum_{1 \leq k \leq n-1} \Delta_k \\
&= n.\Delta_n - \Theta_{n-1} \\
&= n\tfrac{n(n+1)}{2} - \tfrac{(n-1)n(n+1)}{6} \\
&= \tfrac{n(n+1)}{6}(3n - (n-1)) \\
&= \tfrac{n(n+1)}{6}(2n+1) \\
&= \tfrac{n(n+\frac{1}{2})(n+1)}{3}
\end{aligned}
$$

A nice property of pyramid numbers

An interesting question is to compute the sum of two consecutive tetrahedral numbers: $\Theta_n + \Theta_{n-1}$[3].
Recall the first pyramid numbers: $1, 5, 14, 30, 55, \ldots$.

[3] similarly as what we did for $\Delta_n + \Delta_{n-1}$

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It is equal to $\Box_n$

The proof is straightforward by applying the definition:
$\Theta_n = \sum_{k=1}^{n} \Delta_k$

$$\Theta_n + \Theta_{n-1}$$
$$= (\Delta_n + \Delta_{n-1}) + (\Delta_{n-1} + \Delta_{n-2}) + \ldots + (\Delta_2 + \Delta_1) + \Delta_1$$
$$= n^2 + (n-1)^2 + \ldots + 2^2 + 1 = \Box_n$$

[3] similarly as what we did for $\Delta_n + \Delta_{n-1}$

Is there any equivalent of the sum of odd numbers?

We established a strong link between sum of odds $(2n+1)$ and the sum of consecutive triangular numbers.

According to the binomial expression, the next step is to compute the sum of hexagonal numbers $3n^2 + 3n + 1$.
$(n+1)^2 = n^2 + 2n + 1$
$(n+1)^3 = n^3 + 3n^2 + 3n + 1$

Do you think that such numbers are linked with the sum of two consecutive tetrahedral numbers?