Lecture 1 – Maths for Computer Science

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Lecture notes MoSIG1

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Context

The main idea of this preliminary lecture is to **introduce some methodology to prove some results in Discrete Mathematics** (in the field of combinatorics, summations, counting, basic number theory).
We will show how to handle simple and less easy results with very basic tools that do not require any sophisticated background in Maths.

A subsequent goal is to strengthen the intuition while *doing* Maths.

Brief overview of proving techniques

- Contradiction *contradictio in contrarium*
- Induction
- Geometric proofs
- Combinatoric proofs
- Bijections between sets
- Pigeon holes
- All means are good!
- Proofs by computers
- *Fubini's principle*[1].

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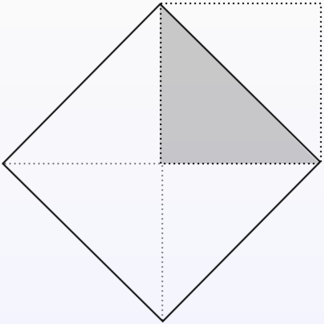
[1]A nice way for proving results on integers is to represent them by sets of *items* (bullets, squares, numbers, intervals). More explanations are coming soon...

Content

Proof by contradiction

Let prove that $\sqrt{2}$ is irrational.



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Assume $\sqrt{2}$ is rational, this means it can be written as $\frac{p}{q}$.
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Thus, $2.q^2 = p^2$.
p^2 is *even* (divisible by 2) then p is also even (the square of an odd number is odd). This means that $p = 2m$ for some positive integer m, which allows us to rewrite:
$2.q^2 = 4.m^2$, after simplification: $q^2 = 2.m^2$.
Thus, q must be even.
Both q and p have a common factor (2), which contradicts the assumption that they both share no common prime divisor.

Proof by induction

Proving that a statement $P(n)$ involving integer n is true.

- **Basis.** Solve the statement for the small values of n.
- **Induction step.** Prove the statement for n assuming it is correct for $k \leq n-1$.

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Solve the following equation:
$U_{n+1} = 2.U_n + 1$, where $U_1 = 1$. Guess $U_n = 2^n - 1$.

Clearly this result holds for $n = 1$.
Assume it is correct for n and write
$U_{n+1} = 2.U_n + 1 = 2.(2^n - 1) + 1 = 2^{n+1} - 1$

A (old and simple) geometrical proof

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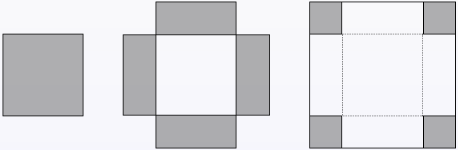

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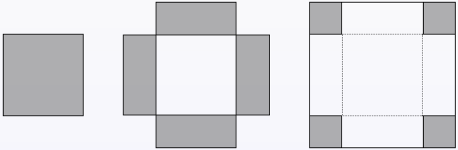

The surface of the cross is equal to the right hand side.
Adding the 4 little squares in the border leads to a total surface of $39 + 4\frac{25}{4} = 64$, which is the square of 8.
We finally deduce x: $8 - 2\frac{5}{2} = 3$.

Pigeon's holes and relations between sets.

The principle here is to establish a correspondence between two sets (pigeons and boxes).
If there are more pigeons than boxes, thus, at least one box contains more than one pigeon.

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Let consider the following problem:
You are attending a party that hosts n couples. In order to create a nice social atmosphere, the hosts requests that each attendees shake the hand of every person that he/she does not know.
Some attendees shake the same number of hands.

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Here, the boxes are the number of times someone shake hands.
The persons are the pigeons. There are $2n$ persons at the party.
The number of people that each attendee does not known is $\{0, 1, ..., 2n-2\}$ which contains $2n-1$ elements.

Proof by computers.

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Coloring planar graphs using no more than 4 colors.
Constraint: 2 neighbor vertices must have different colors.

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Easy to color a planar graph in 6 colors.
For 4 colors, the initial proof needed to check the property on 1478 basic configurations!

Other unconventional ways to prove

The informal idea is to establish a one-to-one correspondence between elements of a set (integers).

Fubini's principle[2]:

Enumerate the elements of a set by two different methods, one leading to an evidence.

[2] Guido Fubini 1879-1943

Content

Triangular numbers

Definition:
Triangular numbers are defined as the sum of the n first integers:
$\Delta_n = \sum_{k=1}^{n} k$.

There exist many proofs for this result, the simplest one is obtained in writing this sum forward and backward and gathering the terms two by two as follows:

$$
\begin{array}{rcccccccc}
2.\Delta_n & = & \boxed{1} & + & \boxed{2} & + & \ldots & + & \boxed{n} \\
 & + & \boxed{n} & + & \boxed{n-1} & + & \ldots & + & \boxed{1}
\end{array}
$$

$$= (n+1) + (n+1) + \ldots + (n+1)$$

$2\Delta_n$ is n times $n+1$, thus, $\Delta_n = \frac{(n+1).n}{2}$

Another way of looking at this process (1)

Use the Fubini's principle.

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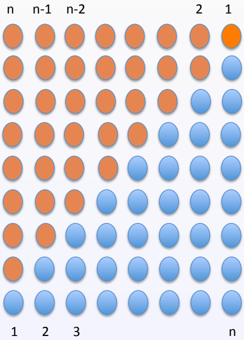
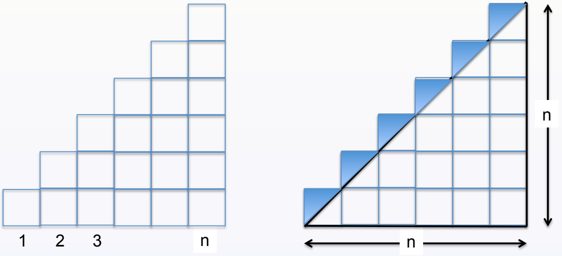
Δ_n is represented by piles of bullets arranged as a triangle. Putting two copies up side down gives the n by $n+1$ rectangle.



Another way of looking at this process (2)

The following figure proves the same result by using a geometric argument instead of bullets.



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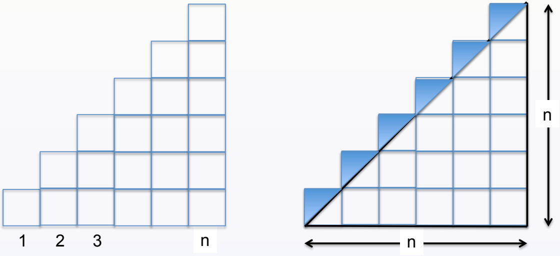

The sum is represented by a tesselation of boxes of size 1 by 1. The global result is determined by the surface of half the big square ($\frac{n^2}{2}$) plus n times half of the surface of the unit square in the diagonal.

Thus, $\frac{n^2}{2} + n.\frac{1}{2} = \frac{(n+1).n}{2}$

Sum of two consecutive triangular numbers

An interesting question is to compute $\Delta_n + \Delta_{n-1}$.
Computing the first ranks leads us to an evidence: $\Delta_1 + \Delta_0 = 1$, then, 4, 9, 16, 25, 36, ...

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It is natural to guess $\Delta_n = n^2$, which is easy provable by induction (or alternatively, using the expression $\Delta_n + \Delta_{n-1} = n + 2\Delta_{n-1}$ since $\Delta_n = n + \Delta_{n-1}$).

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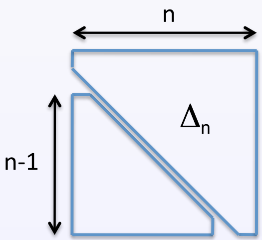
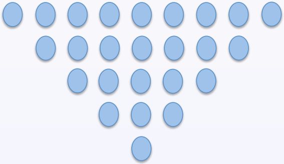
This result can be directly obtained using a geometric pattern:

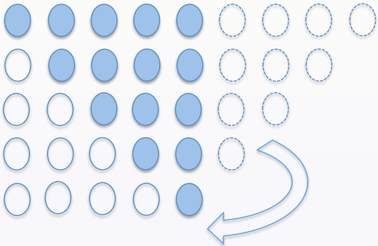


Content

Sum of odds

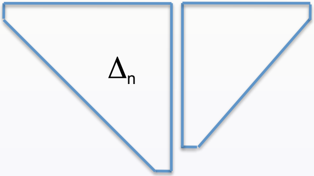
Determine the sum of the first odd integers, denoted by $S_n = \sum_{k=0}^{n-1}(2k+1)$.
This result may again be established by using Fubini's principle.
The bullets depict the consecutive odd numbers. The arrangement of the bullets gives two ways for counting.





$$S_n = n^2$$

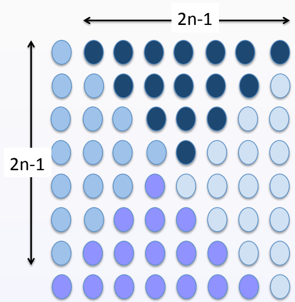
Rearranging the figure as a butterfly gives an insight of another expression of $\Delta_n + \Delta_{n-1}$.



The analytic expression is as follows:

$$\begin{aligned}\Delta_n + \Delta_{n-1} &= 1 + 2 + 3 + \dots + n \\ & + 1 + 2 + \dots + n-1 \\ &= 1 + 3 + 5 + \dots + (2n-1)\end{aligned}$$

We can also imagine an alternative construction which uses four copies of S_n that exactly correspond to an $2n$ by $2n$ square as depicted in the figure.



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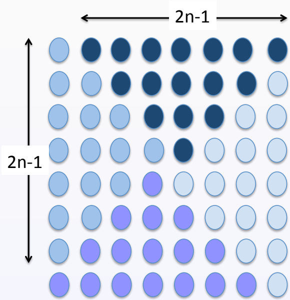
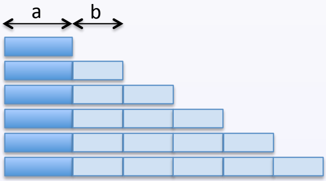


This leads to $4.S_n = (2n)^2$, thus $S_n = n^2$.

Generalization (1)

Both previous examples of triangular and sum of odd numbers are special cases of arithmetic progressions: starting at $p_1 = a$, $p_n = p_{n-1} + b$ for $n > 2$.

The figure below shows that the sum of the n first elements of an arithmetic progression is equal to $n.a + \Delta_{n-1}.b$.



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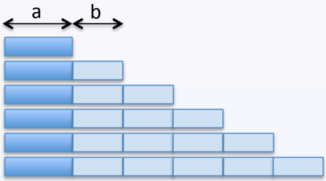

For instance, $a = 1$ and $b = 2$ for the sum of the first n odd numbers. We have $S_n = n + 2.\Delta_{n-1} = n + n(n-1) = n^2$.

Generalization (2)

Another interesting case is for $a = 1$ and $b = 4$ (1, 5, 9, 13, 17, ...).

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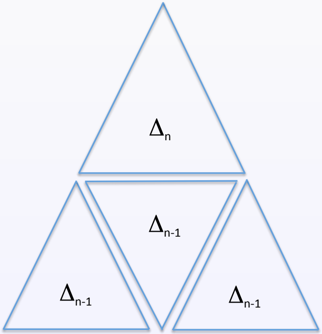
Another interesting case is for $a = 1$ and $b = 4$ (1, 5, 9, 13, 17, ...).
The sum is equal to $n(2n - 1) = n + 4\Delta_{n-1}$.

$= \Delta_n + 3\Delta_{n-1}$ which is also a triangular number (Δ_{2n-1}).



Various ways to solve the sum of squares

Definition:
Sum of the n first squares:
$\square_n = \sum_{k=1}^{n} k^2$.

Method 1: determine the asymptotic behavior

Very rough analysis:
as $k^2 \leq n^2 \; \forall k \leq n$, $\square_n \leq \sum_{k=1}^{n} n^2 = n^3$.

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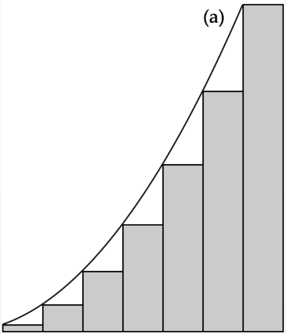
Very rough analysis:
as $k^2 \leq n^2 \; \forall k \leq n$, $\square_n \leq \sum_{k=1}^{n} n^2 = n^3$.
A slightly more precise analysis is: $\square_n \leq c\frac{n^3}{3}$



In other words, it is in $O(\frac{n^3}{3})$.

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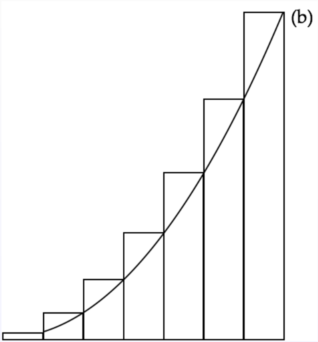
$\square_n \geq c' \frac{n^3}{3}$



It is in $\Omega(\frac{n^3}{3})$, thus, it is $\Theta(\frac{n^3}{3})$

Method 2: by induction

Compute the first ranks:

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	9	16	25	36	49	64	81	100
Sn	0	1	5	14	30	55	91	140	204	285	385

Guess the expression (or take it in a book):
$\square_n = \frac{n(n+1)(2n+1)}{6}$

Strong induction

- Basis: $\square_1 = \frac{(2\times 3)}{6} = 1^2$
- Assume $\square_n = \frac{n(n+1)(2n+1)}{6}$

  Compute $\square_{n+1} = \square_n + (n+1)^2$

  $= (n+1)\frac{n(2n+1)}{6} + n + 1$

  $= (n+1)\frac{2n^2+n+6n+6}{6}$

  $= \frac{(n+1)(n+2)(2n+3)}{6}$

Method 3: undetermined coefficients

Let write $\square_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3$

$\square_0 = \alpha_0 = 0$
$\square_1 = \alpha_1 + \alpha_2 + \alpha_3 = 1$
$\square_2 = 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = 5$
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$\alpha_1 = \frac{1}{6}$, $\alpha_1 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{3}$

Thus, $\square_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$

Method 4: perturb the sum

Developing two ways to compute $C_n = \sum_{k=1}^{n} k^3$ allows to express $\square_n$.

1. $C_{n+1} = 1 + \sum_{k=2}^{n+1} k^3$

   $= 1 + \sum_{k=1}^{n} (k+1)^3$

   $= 1 + \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1)$

   $= 1 + C_n + 3\square_n + 3\Delta_n + n$

2. $C_{n+1} = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n+1)^3 + C_n$

   $= n^3 + 3n^2 + 3n + 1 + C_n$

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   $= 1 + C_n + 3\square_n + 3\Delta_n + n$
2. $C_{n+1} = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n+1)^3 + C_n$
   $= n^3 + 3n^2 + 3n + 1 + C_n$

Let now equal both expression to deduce $\square_n$.

$$1 + 3\square_n + 3\frac{n^2+n}{2} + n = n^3 + 3n^2 + 3n + 1$$

$$3\square_n = n^3 + 3n^2 + 2n - 3\frac{n^2+n}{2} = n^3 + \frac{3n^2}{2} + \frac{n}{2}$$

Method 5: expand and contract the sum

$\square_n = \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$

$= 1 + (2 + 2) + (3 + 3 + 3) + (4 + 4 + 4 + 4) + \ldots + (n + n + \ldots + n)$

Method 5: expand and contract the sum

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$$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$$

$$= 1 + (2+2) + (3+3+3) + (4+4+4+4) + ... + (n+n+...+n)$$

$$= (1+2+....+n) + (2+3+...+n) + (3+4+...+n) + ... + n$$

$$= \sum_{k=0}^{n-1} (\Delta_n - \Delta_k)$$

$$= n.\Delta_n - \sum_{k=1}^{n-1} \Delta_k$$

$$\square_n = \frac{n^2(n+1)}{2} - \sum_{k=1}^{n-1} \frac{k^2}{2} - \frac{1}{2}\Delta_{n-1}$$

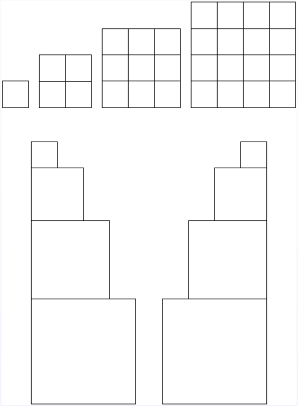
$$\square_n = \frac{n^2(n+1)}{2} - \frac{1}{2}(\square_n - n^2) - \frac{n(n-1)}{4}$$

$$\frac{3}{2}\square_n = \frac{1}{2}(n^3 + n^2 + n^2 - \frac{n^2-n}{2})$$

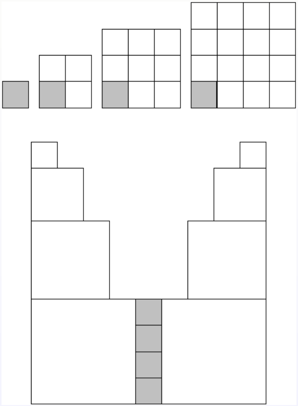
$$\square_n = \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{n}{2})$$

Method 6: graphical proof

Consider 3 copies of the sum represented by unit squares.



Graphical proof



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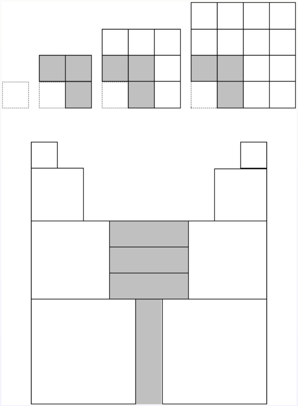

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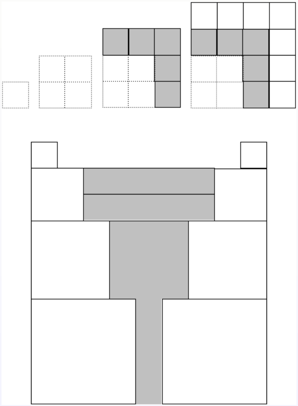

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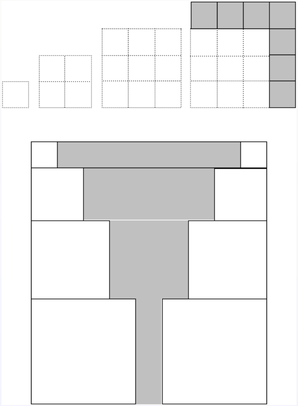

Graphical proof

Conclusion: The area of the 3 sums is equal to a big rectangle $2n+1$ by $\Delta_n = \frac{n(n+1)}{2}$.

Thus, $3\Box_n = \frac{(2n+1)n(n+1)}{2}$