Lecture 5 and 6 – Maths for Computer Science Discrete Structures

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Lecture notes MoSIG1

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Objective and Plan

The purpose of this lecture is to present discrete structures and related ways to operate on such structures, mainly *Graphs*.

Graphs

Set of *objects* in relation.

Many examples in the *real life*

Part 1 deals with the definitions and the analysis of properties on structured graphs.
Part 2 is dedicated to how to use graphs for solving problems.

What is a graph?

A graph is a finite set of vertices linked according to a given relation.

Formally

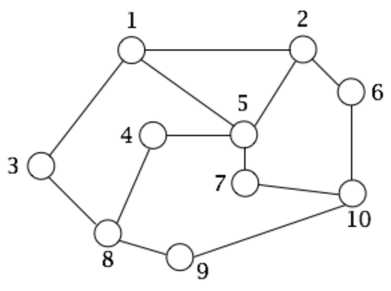
A graph G is defined as the pair (V, E).
V is the (finite) set of vertices.
The set of edges E represents the relations between all pairs of vertices.

There is a large variety of graphs:
directed graphs, weighted graphs, multiple graphs, hyper-graphs, etc..

How to represent graphs?

- Picturally
- Algebraically (Adjacency matrices)

An example.

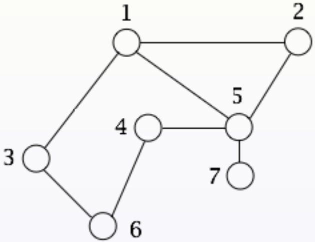


Adjacency matrices

In the previous graph:
vertex 1 is linked with 2, 3 and 5
vertex 2 is linked with 1, 5 and 6
vertex 3 is linked with 1 and 8
...

These relations are represented by a matrix as follows:
$A_{i,j} = 1$ if vertex i is linked with vertex j $(i \neq j)$
$A_{i,j} = 0$ otherwise

Adjacency matrices



$$\mathbf{A} = \begin{array}{|ccccccc|} \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Interest of the Algebraic approach

How to determine if a graph is fully connected?

Compute A^2.

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Compute A^2.

$$A^2 = \begin{bmatrix} 3 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The element in row i and column j gives the number of paths of length exactly equal to 2.

Graph Characteristics

Several notions tell us about the graph structures:

- Degree
- Diameter
- Chromatic index

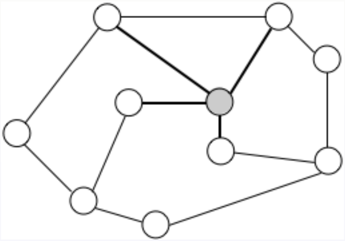
The last characteristic is hard to compute while both others are easy to obtain (in polynomial time).

Degree

Definition

The degree of a vertex x is the number of vertices adjacent to x.
Denoted by $\delta(x)$ for $x \in V$

The maximum (resp. average) degree of a graph is the maximum of the degree of its vertices (resp. average).
A graph where all the degrees are the same is *regular*.



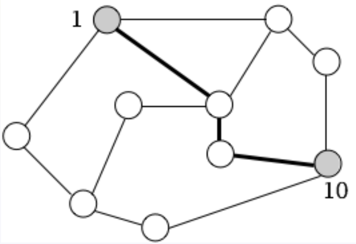
The degree of the shaded vertex is 4.

Diameter

We define a *natural distance* between two vertices as the minimum number of edges to cross for connecting both vertices.

Definition

maximum distance between any pairs of vertices denoted by D



The length of the path between the two shaded vertices is 3.
The Diameter of the graph is also 3.

Path is graph

Definitions

A path is sequence of edges such that two adjacent edges share a vertex.
The length of the path is the number of edges of the sequence.

Graph connectivity

We introduce the relation x **is connected to** y if there exists a path between x and y.

Property

The connectivity is an equivalence relation.

It induces connected components.

The distance can be extended easily to weighted graphs.

Extension to directed graphs

Here, a path takes into account the direction of the *arcs*.

Strongly connected relation

There exists a path between x and y and a path between y and x.

An important property

Statement

In any undirected graph, the number of the vertices with odd degrees is even.

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Proof

$\sum_{x \in V} \delta(x) = 2.|E|$
Thus, $\sum_{x \in V} \delta(x)$ is even
$= \sum_{x \in Vodd} \delta(x) + \sum_{x \in Veven} \delta(x)$
and obviously, $\sum_{x \in Veven} \delta(x)$ is even,
thus $\sum_{x \in Vodd} \delta(x)$ is even, and necessary, their number is even.

Chromatic index

Two adjacent vertices are colored in a different color.

Definition

minimum number of colors for coloring the graph.

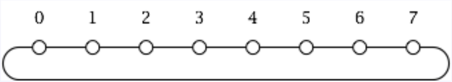
Rings or Cycles

Definition.

Each vertex of C_n has exactly one predecessor and one successor.

Coding of edges

$$\big\{\{i,\ i+1 \bmod n\} \ \mid \ i \in \{0,\ 1,\ \ldots,\ n-1\}\big\}.$$



Properties.

Connected (diameter $\lceil \frac{n}{2} \rceil$) and regular graph (degree 2).

Complete graphs (or cliques)

Definition.
Each vertex of K_n is connected to all the other vertices.

- Connected ($D = 1$)
- Regular graph ($\delta = n - 1$)
- Number of edges $\frac{(n)(n-1)}{2}$

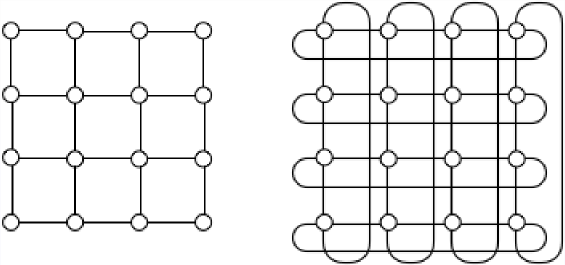
Meshes and Torus

Definition.
Cartesian product of cycles.

Coding (vertices and edges)).

$$\{1,\ 2, \ldots,\ m\} \times \{1,\ 2, \ldots,\ n\}$$
$$\Big\{\langle i,\ j\rangle \ \mid\ \big[i \in \{1,\ 2, \ldots,\ m\}\big],\ \ \big[j \in \{1,\ 2, \ldots,\ n\}\big]\Big\}$$

Example for $n = 4$



Properties of the square torus

$\sqrt{n}$ by $\sqrt{n}$

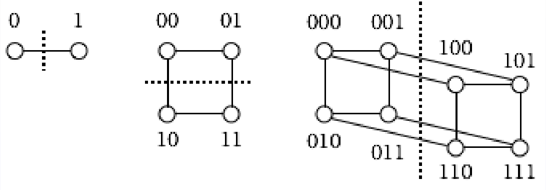
- Connected (diameter $\sqrt{n}$)
- Regular graph (degree 4)
- Number of edges: $2n$

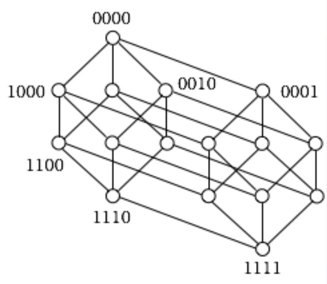
Hypercubes

Recursive Definition.

- The order-0 boolean hypercube, H_0, has a single vertex, and no edges.
- The order-$(k+1)$ boolean hypercube, H_{k+1}, is obtained by taking two copies of H_k ($H_k^{(1)}$ and $H_k^{(2)}$), and creating an edge that connects each vertex of $H_k^{(1)}$ with the corresponding vertex of $H_k^{(2)}$.

Construction




0000
1000
0010
0001
1100
1110
1111

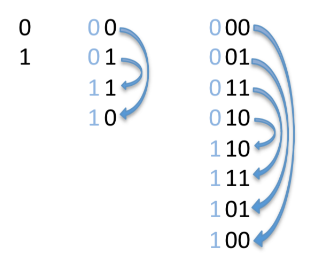
Coding

A natural binary coding

There exist several variants of Gray codes. The most important characteristic is: the coding from one position to the next is to flip only one bit.
Let us present the most popular one **Reflected Gray code**

The 1-bit Gray code is simply 0 and 1.
The next one (for 2-bits) is obtained by mirroring the 1-bit code and prefix it by 0 and 1.
The next ones are obtained similarly

Gray Code



How to obtain the Gray code from the binary code?

00001	00001
00010	00011
00011	00010
00100	00110
00101	00111
00110	00101
00111	00100
01000	01100
01001	01101
01010	01111
01011	01110
01100	01010
01101	01011
...	...

Characteristics of Hypercubes

The number of vertices is a power of 2: $n = 2^k$ $(k = log_2(n))$

- Degree $\delta_n = k$
- Diameter $\delta_n = k$
- Number of edges?

Characteristics of Hypercubes

The number of vertices is a power of 2: $n = 2^k$ $(k = log_2(n))$

- Degree $\delta_n = k$
- Diameter $\delta_n = k$
- Number of edges?
  H_{k+1} is obtained by two copies of H_k plus 2^k edges for linking each relative vertex, thus:
  $N_{k+1} = 2 \times N_k + 2^k$ starting at $N_0 = 0$
  $N_k = k \times 2^{k-1}$

Graph isomorphism

The question here is that there are many ways to draw a graph...

The hypercube H_4 is identical to the 4 by 4 torus.

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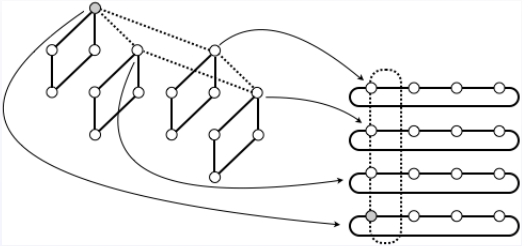
The hypercube H_4 is identical to the 4 by 4 torus.
The proof is by an adequate coding of the vertices/edges.



Matching

Definition

A matching is a set of edges that have no vertices in common.

It is *perfect* if its vertices are all belonging to an edge[1].

Proposition.

The number of perfect matchings in a graph of order $n = 2k$ grows exponentially with k.

[1] thus, the number of vertices is even and the cardinality of the matching is exactly half of this number

Proof

by recurrence on k,
let denote the number of perfect matchings by N_k.

Base case: For $k = 1$, there is only one perfect matching $N_1 = 1$ and for $k = 2$, there are 3 different perfect matchings $N_2 = 3$.
Induction step: For k, there are $2k - 1$ possibilities for a vertex to choose an edge, $N_k = (2k - 1).N_{k-1}$
Thus, N_k is the product of the k first odd numbers.

However, determining a perfect matching of minimal weight in a weighted graph can be obtained in polynomial time (using the Hungarian assignment algorithm).

Trees

Definition

Connected graphs without cycles.

Trees

Definition

Connected graphs without cycles.

Proposition

Any tree of order n has specifically $n-1$ edges.

A preliminary result

Lemma

Any connected tree of order n ($n \geq 1$) has at least one vertex of degree 1 (called a leaf).

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Rapid Proof

The main argument is on the analysis of graphs with minimum degrees 0 and 2.

Proof

Principle

By induction on the order of the graph n.

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By induction on the order of the graph n.

Base case for $n = 2$
Induction step. Use the previous Lemma.

Spanning Trees

Let consider a weighted graph G.

Motivation

a way of succinctly summarizing the connectivity structure inherent in undirected graphs.

Definition

Take the same set of vertices and extract a set of edges that spans the vertices.

MST: Determine a minimal Spanning Tree (easy problem, two possible philosophies).

Planar and outer-planar

Definition 1

A graph is planar if it can be drawn without any crossing edges.

Definition 2

A graph is outer-planar if it can be drawn by placing its vertices along a circle in such a way that its edges can be drawn as non-crossing chords of the circle.

Proposition

Every Tree is outer-planar.

Examples

- K_3 is planar
- K_4 is planar but not outer-planar
- K_5 is not planar

- $K_{1,3}$ is outer-planar
- $K_{2,3}$ is planar but not outer-planar
- $K_{3,3}$ is not planar

The 3-coloring theorem of outer-planar graphs.

Proposition

Every outer- planar graph is 3-colorable.

The proof is by induction on the number of vertices n.