Lecture 3 – Maths for Computer Science Solving recurrences and Fibonacci numbers

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Objective and plan

The purpose of this lecture is to go deeper into recurrence proofs, in particular, bilinear recurrences.

$u_{n+1} = \alpha.u_n + \beta.u_{n-1} + \gamma$ where u_0 and u_1 are given.

Applications

We already studied such an inductive expression.

Token Game

$T(n+1) = T(n) + 2.T(n-1) + 1$ with $T(0) = 1$ and $T(1) = 2$

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Fibonacci numbers

The simplest possible bilinear recurrence ($\alpha = \beta = 1$ and $\gamma = 0$).
$F(n+1) = F(n) + F(n-1)$ with $F(0) = 1$ and $F(1) = 1$

Lucas' numbers

Same as Fibonacci with a different seed.
$L(n+1) = L(n) + L(n-1)$ with $L(0) = 1$ and $L(1) = 3$

Derangements

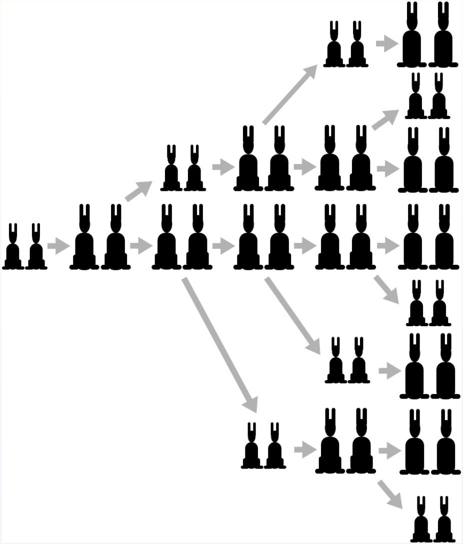
$d(n+1) = n\,(d(n-1) + d(n-2))$ with $d(0) = 1$ and $d(1) = 2$

Definition of Fibonacci numbers

The original problem has been introduced by Leonardo of Pisa (Fibonacci) in the middle age.

- Fibonacci numbers are the number of pairs of rabbits that can be produced at the successive generations.
- Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition (pictorially)



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Given the two numbers $F(0) = 1$ and $F(1) = 1$
the Fibonacci numbers are obtained by the following expression:
$F(n+1) = F(n) + F(n-1)$

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The first ranks:

n	0	1	2	3	4	5	6	7	8	9	10	...
F(n)	1	1	2	3	5	8	13	21	34	55	89	...

Combinatorial interpretation

Proposition

The Fibonacci number $F(n)$ can be interpreted as the number of length-n binary strings in which each occurence of a 1 is directly preceded by a 0.

Let S_n be the set of such strings of length n.

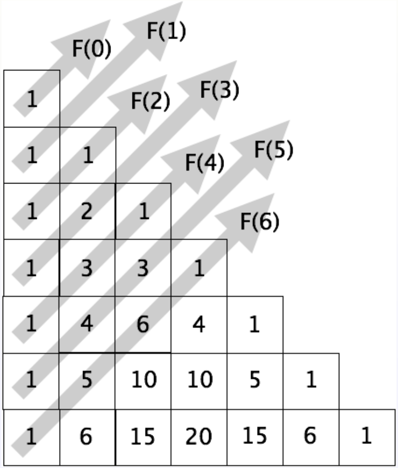
Proof

By the previous definition, every binary string ω_n ends either with 0 or with 01.

- If ω_n ends with 0, then, it has the form $x0$ where the prefix x is a binary string of length $n-1$.
  Moreover, x must belongs to S_{n-1} in order ω_n belongs to S_n.
  Therefore S_n contains $|S_{n-1}|$ strings of this form.
- If ω_n ends with 01, then it has the form $\omega_n = y01$, where the prefix y is a binary string of length $n-2$.
  Moreover, y must belong to S_{n-2} in order for ω_n to belong to S_n, that contains $|S_{n-2}|$ strings of this form.

$$F(n) = |S_n| = F(n-1) + F(n-2)$$

Link with the Pascal's triangle



Studying a first property

Proposition:

$F(n+2) = 1 + \sum_{k=0}^{n} F(k)$

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Let check the expression on the first ranks:

$n = 1$,
$F(3) = 1 + F(1) + F(0) = 1 + 1 + 1 = 3$

$n = 2$,
$F(4) = 1 + F(2) + F(1) + F(0) = 1 + 2 + 1 + 1 = 5$

$n = 3$,
$F(5) = 1 + F(3) + F(2) + F(1) + F(0) = 1 + 3 + 2 + 1 + 1 = 8$
...

Proof

By induction

- The **basis case** (for $n = 0$) is true since $F(2) = 1 + F(0)$.
- **Induction step:** Let assume the property holds at rank n for $F(n+2)$ and compute $F(n+3)$:
  Apply the definition of Fibonacci numbers:
  $F(n+3) = F(n+1) + F(n+2)$
  Replace the last term by the recurrence hypothesis:
  $F(n+2) = 1 + \sum_{k=0}^{n} F(k)$
  Thus,
  $F(n+3) = F(n+1) + 1 + \sum_{k=0}^{n} F(k) = 1 + \sum_{k=0}^{n+1} F(k)$

Product of two consecutive Fibonacci numbers

Proposition:

$F(n).F(n-1) = \sum_{k=0}^{n-1} F(k)^2$ (for $n \geq 1$)

Let check the expression on the first ranks:

$n = 2,\ F(2).F(1) = F(1)^2 + F(0)^2 = 1 + 1 = 2$

$n = 3,\ F(3).F(2) = F(2)^2 + F(1)^2 + F(0)^2 = 4 + 1 + 1 = 6$

$n = 4,\ F(4).F(3) = F(3)^2 + F(2)^2 + F(1)^2 + F(0)^2 = 15$

$n = 5,\ F(5).F(4) = F(4)^2 + F(3)^2 + F(2)^2 + F(1)^2 + F(0)^2 = 40$

...

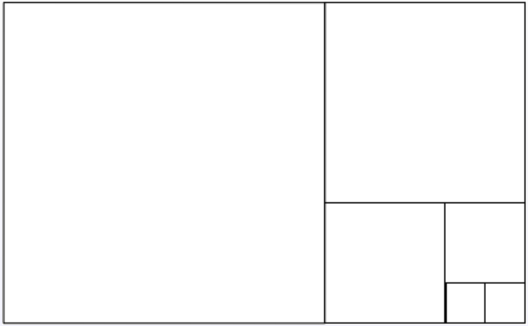
Proof by induction

- The **basis case** (for $n = 1$) is true since $F(1).F(0) = F(0)^2 = 1$.
- **Induction step**[1]**:** Let assume the property holds at rank n and compute $F(n+1).F(n)$:
  Apply the definition of $F(n+1)$:
  $F(n+1).F(n) = (F(n) + F(n-1)).F(n)$
  $= F(n)^2 + F(n).F(n-1)$
  Apply now the induction hypothesis to this last term:
  $F(n+1).F(n) = F(n)^2 + \sum_{k=0}^{n-1} F(k)^2 = \sum_{k=0}^{n} F(k)^2$

[1] exactly the same scheme as before!

An alternative proof by recurrence

The relation can be proved very easily by the geometric argument shown below



Another property dealing with squares

Proposition:

$F(n+2)^2 = 4.F(n).F(n+1) + F(n-1)^2$ for $n \geq 2$.

Let check the expression on the first ranks:

$n = 1,\ F(3)^2 = 3^2 = 4.F(1).F(2) + F(0)^2 = 8 + 1 = 9$

$n = 2,\ F(4)^2 = 5^2 = 4.F(2).F(3) + F(1)^2 = 24 + 1 = 25$

$n = 3,\ F(5)^2 = 8^2 = 4.F(3).F(4) + F(2)^2 = 60 + 4 = 64$

$n = 4,\ F(6)^2 = 13^2 = 4.F(4).F(5) + F(3)^2 = 160 + 9 = 169$

...

Analytic proof

Use the definition of the Fibonacci numbers and expand:

$$F(n+2)^2 = (F(n+1) + F(n))^2$$

$$= F(n+1)^2 + 2.F(n+1).F(n) + F_n^2$$

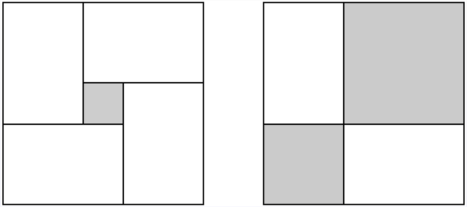
$$= 4.F(n+1).F(n) - 2.F(n+1).F(n) + F(n+1)^2 + F(n)^2$$

$$= 4.F(n+1).F(n) + (F(n+1) - F(n))^2$$

Again, using the definition of $F(n+1)$ into the square, we get the expected result:

$$F(n+2)^2 = 4.F(n+1).F(n) + F(n-1)^2$$

Graphical proof



Cassini's identity

Proposition:

$F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1}$ for $n \geq 1$

Let check the expression on the first ranks:

$n = 1,\ F(0).F(2) = F(1)^2 + 1 = 2$

$n = 2,\ F(1).F(3) = F(2)^2 - 1 = 3$

$n = 3,\ F(2).F(4) = F(3)^2 + 1 = 10$

$n = 4,\ F(3).F(5) = F(4)^2 - 1 = 24$

...

Proof (by induction)

- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini's identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n{+}2) = F(n)(F(n{+}1){+}F(n)) = F(n)^2{+}F(n).F(n{+}1)$

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  Apply the definition of $F(n+2)$:
  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n)^2+F(n).F(n+1)$
  Replace the last term using the recurrence hypothesis:
  $F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1}$
  $= F(n-1).F(n+1) + (-1)^{n+2}$

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  Thus,
  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$
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  Thus,
  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$
  $= F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$
  Apply again the definition of Fibonacci sequence
  $F(n) + F(n-1) = F(n+1)$, we obtain:
  $F(n).F(n+2) = F(n+1)^2 + (-1)^{n+2}$

Computing $F(n)$ fast

$F(n)$ can be computed in $log_2(n)$ steps.

Proposition.

For all integers n:
(a) $F(2n) \;=\; (F(n))^2 \;+\; (F(n-1))^2$;
(b) $F(2n+1) \;=\; F(n) \times (2F(n-1) + F(n))$.

Details (a) – Proof by induction

The base case $n = 1$ is true because

$$\begin{aligned} F(2) &= (F(1))^2 + (F(0))^2 = 2 \\ F(3) &= F(1) \times (2F(0) + F(1)) = 3 \end{aligned}$$

Assume that the property holds for n, for both $F(2n)$ and $F(2n+1)$.

$$F(2(n+1)) = F(2n+1) + F(2n)$$

$$= (F(n))^2 + (F(n-1))^2 + F(n) \times (2F(n-1) + F(n))$$

$$= (F(n))^2 + (F(n-1))^2 + 2(F(n) \times F(n-1)) + (F(n))^2$$

$$= (F(n) + F(n-1))^2 + (F(n))^2$$

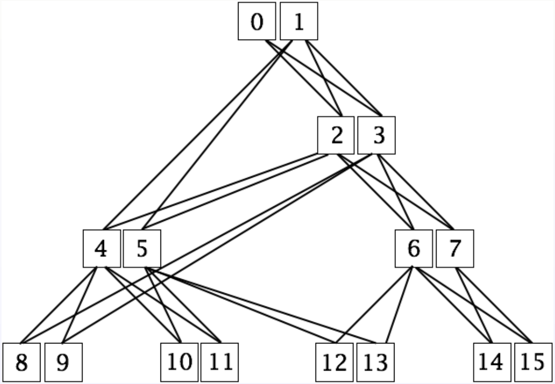
$$= (F(n+1))^2 + (F(n))^2$$

Details (b)

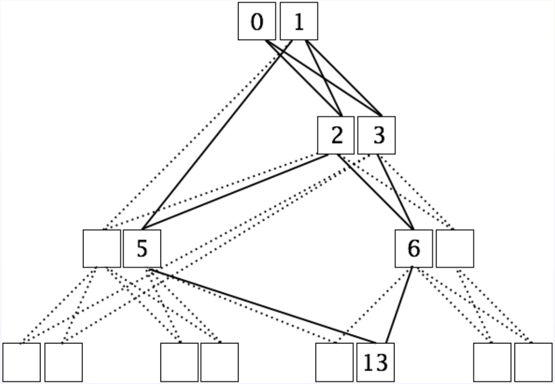
We again start by applying the defining recurrence of the Fibonacci numbers on $F(2(n+1)+1)$

$$= F(2(n+1)) \; + \; F(2n+1)$$
$$= (F(n+1))^2 \; + \; F(n)^2 \; + \; F(n) \times (2F(n-1) + F(n))$$
$$= (F(n+1))^2 \; + \; 2(F(n-1) + F(n)) \times F(n)$$
$$= (F(n+1))^2 \; + \; 2F(n+1) \times F(n)$$

Pictorially



Pictorially (from one node)



Definition of Lucas' numbers

A natural question is:

what happens if we change the first ranks of the sequence keeping the same recurrence pattern?

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It has been studied by the french mathematician Edouard Lucas, starting at 2 and 1 .

For some reasons that will be clarified later, the sequence is shifted backwards (we take the convention $L(-1) = 2$).

Definition of Lucas' numbers

Definition:
Given the two numbers $L(0) = 1$ and $L(1) = 3$
all the other Lucas' numbers are obtained by the same progression as Fibonacci:

- $L(n + 1) = L(n) + L(n - 1)$

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n	-1	0	1	2	3	4	5	6	7	8	9	10
F(n)		1	1	2	3	5	8	13	21	34	55	...
L(n)	2	1	3	4	7	11	18	29	47	76	123	...

- There are[2] strong links with Fibonacci numbers.

In particular, we established before that:

$F(n+2) = 1 + \sum_{k=0}^{n} F(k)$.

We have similarly:

$L(n+2) = 1 + \sum_{k=-1}^{n} L(k)$
since the basic step of the induction is still valid[3].
$L(2) = L(-1) + L(0) + 1 = 2 + 1 + 1 = 4$.

[2] of course
[3] It will be true for all the progressions where $u_1 = 1$

A first Property

We can also easily show that the Lucas number of order n is the sum of two Fibonacci numbers:

Proposition.

$L(n) = F(n-1) + F(n+1)$ for $n \geq 1$

Let *check* this property on the first ranks:

$n = 2,\ L(2) = F(1) + F(3) = 1 + 3 = 4$

$n = 3,\ L(3) = F(2) + F(4) = 2 + 5 = 7$

$n = 4,\ L(4) = F(3) + F(5) = 3 + 8 = 11$

$n = 5,\ L(5) = F(4) + F(6) = 5 + 13 = 18$

...

Proof by induction

- The **basis case** (for $n = 1$) is true since
  $L(1) = 3 = F(2) + F(0) = 2 + 1$.
- **Induction step:** Let assume the property holds at all ranks
  $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$

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- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$

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  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$

Proof by induction

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- **Induction step:** Let assume the property holds at all ranks
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  Apply the definition of Lucas' numbers:
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  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for
  $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$
  replace them in the previous expression:
  $L(n+1) = F(n+2) + F(n)$

which concludes the proof.

Extension 1

Notice that using a similar approach, we obtain $L(n) = F(n+2) - F(n-2)$.

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What happens if we generalize?

Proposition.

$2.L(n) = F(n+3) + F(n-3)$

Proof.

We start from $L(n) = F(n+2) - F(n-2)$
$F(n+2) = F(n+3) - F(n+1)$ and
$F(n-2) = F(n-1) - F(n-3)$
$L(n) = F(n+3) - (F(n+1) + F(n-1)) + F(n-3)$
$2.L(n) = F(n+3) + F(n-3)$

Extension 2

Go to the next step using the same technique:

$2.L(n) = F(n+3) + F(n-3)$
$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$

$3.L(n) = F(n+4) - F(n-4)$

[4] The formal proof is let to the reader

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$$3.L(n) = F(n+4) - F(n-4)$$

One more step: $5.L(n) = F(n+5) + F(n-5)$

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$$2.L(n) = F(n+3) + F(n-3)$$
$$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$$

$$3.L(n) = F(n+4) - F(n-4)$$

One more step: $5.L(n) = F(n+5) + F(n-5)$

Thus, we guess the following expression.

Proposition[4].
$F(k-1).L(n) = F(n+k) + (-1)^{k-1}F(n-k)$ for $k \leq n$

[4] The formal proof is let to the reader

Two other propositions

Proposition.
$F(n+1) = \frac{1}{2}(F(1).L(n) + F(n).L(1))$

The proof comes from direct arithmetic manipulations:
$2.F(n+1) = F(n+1) + F(n+1)$

$= F(n+1) + F(n) + F(n-1)$

$= L(n) + F(n)$

$= F(1).L(n) + F(n).L(1)$

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$= F(n+1) + F(n) + F(n-1)$

$= L(n) + F(n)$

$= F(1).L(n) + F(n).L(1)$

The previous property can be extended for any $k > 1$
Let compute the expression of $F(k).L(n) + F(n).L(k)$

A final natural question

The golden ratio.

It is a well-known result that the ratio of two consecutive Fibonacci number tends to the Golden ratio:
$lim_{n\to\infty}\frac{F(n)}{F(n-1)} = \Phi$

As this result is obtained by solving the following equation $x^2 = x + 1$ (Φ is the positive root) and does not depend on the first rank, this holds also for the Lucas' numbers.

A last result: the Zeckendorf's Theorem

Objective: Study the Fibonacci numbers as a numbering system.

Let us first introduce a notation: $j \gg k$ iff $j \geq k + 2$.
The *Zeckendorf's theorem* states that:

every positive integer n has a unique decomposition of the form:
$n = F_{k_1} + F_{k_2} + ... + F_{k_r}$ where $k_1 \gg k_2 \gg ... \gg k_r$ and $k_r \geq 2$

Here, we assume that the Fibonacci sequence starts at index 1 and not 0, moreover, the decompositions will never consider $F(1)$ (since $F(1) = F(2)$).

Derangements

Derangements represent one of the simplest forms of *avoidance problems*.

- A professor views it as a win-win strategy for the students in her class to grade each others' essays on *The Essential Truth in the Universe*.
  The essays thereby get graded faster.
- Moreover, each student gets a chance to see how another student has interpreted some basic component of the human experience.
- The only complication is: How should we allocate essays among the students?

The process must ensure that no student is assigned her own essay to critique.

This challenge is known as a *derangement problem*.

Derangements

- A *derangement* of a (finite) set A is a *bijection* $f : A \leftrightarrow A$ that has no *fixed point*.

In other words, for every $a \in A$, we must have $f(a) \neq a$.

Clearly, derangements always exist (for $n > 1$).

One can just label the elements of set A by the numbers $0, 1, \ldots, |A| - 1$ and specify $f(a) = a + 1 \bmod |A|$.

Playing around with a simple example

However, derangements are not so common! In fact, the set $A = \{0, 1, 2\}$ admits six self-bijections, but only two are derangements. Which ones?

Playing around with a simple example

However, derangements are not so common! In fact, the set $A = \{0, 1, 2\}$ admits six self-bijections, but only two are derangements. Which ones?

$$f(a) = a + 1 \bmod 3 \quad : \text{which maps} \quad (0 \to 1), (1 \to 2), (2 \to 0)$$

$$g(a) = a - 1 \bmod 3 \quad : \text{which maps} \quad (0 \to 2), (1 \to 0), (2 \to 1)$$

- How many derangements does an arbitrary n-element set A have? We denote this quantity by $d(n)$.

Derangements

We compute $d(n)$ for arbitrary integer n via the following recursion:

- For $n = 1$: $d(1) = 0$.

  The unique bijection in this case consists only of a fixed point.
- For $n = 2$: $d(2) = 1$.

  There are two bijections in this case
  - the identity, which has two fixed points
  - the swap, which is a derangement.

The inductive expression

- For $n > 2$: $d(n) = (n-1)(d(n-1) + d(n-2))$:

  To see this, note first that in any derangement, the first element of A, call it a, must map to some $b \neq a$.
  - Note next that there are $n-1$ ways to choose b.
  - There are $d(n-2)$ derangements under which b maps to a. In those cases, we know everything about a and b, so we need worry only about the remaining elements of A. These $n-2$ elements can "derange" in all possible ways.
  - There are $d(n-1)$ derangements under which element b does not map to a.

An observation

The preceding reasoning verifies the following recurrence

$$d(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ n(d(n-1) + d(n-2)) & \text{if } n > 2 \end{cases}$$

Solving the recurrence

There are several ways to solve this recurrence.

- We can reduce the bilinearity by a linear recurrence:

$$d(n) = \begin{cases} 0 & \text{if } n = 1 \\ n \cdot d(n-1) + (-1)^n & \text{if } n > 1 \end{cases}$$

Interestingly, as the number of objects in the set to be deranged grows without bound.

The proportion of bijections that are derangements tends to the limit $1/e$, where e is the base of natural logarithms.

Concluding remarks and take home message

We are more interested here by the dynamic process of *computing* the successive terms of the progressions than in their asymptotic limits when n is large...