Mathematics for Computer Science, mid-term short exam

Elements of solutions and comments

Jean-Marc.Vincent@univ-grenoble-alpes.fr

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This presentation is not a correction of the exam, it provides elements of solutions (which are not unique) and some comments that we suppose helpful for next exams.

The context

Important information.
Read this before anything else !

- Documents are not allowed
- Electronic devices are not allowed
- The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- All answers should be well-argued to be considered correct.
- All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- Question during the exam : if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

FIBONACCI NUMBERS

Question

Prove the following expression by recurrence (known as Cassini identity) :

$$F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1} \text{ for } n \geqslant 1 \quad (1)$$

where $F(n)$ is the n-th term of the Fibonacci progression $F(0) = F(1) = 1$

Method (on your draft)

- **Context of the question** Identity on Fibonacci numbers defined by a recurrence equation

$$F(n) = F(n-1) + F(n-2) \text{ with } F(0) = F(1) = 1 \quad (2)$$

- **Aim** The aim is to prove the identity using a recurrence proof.
- Recurrence The proof is decomposed in two parts, the base case and the induction property.
  1. State the property $\mathcal{P}(n)$ and the possible values of n
  2. Base case : prove that $\mathcal{P}(1)$ (starting point depends of the problem)
  3. Induction : prove that for any n if $\mathcal{P}(n)$ then $\mathcal{P}(n+1)$, that is

$$\forall n \geqslant 1 \quad \mathcal{P}(n) \Longrightarrow \mathcal{P}(n+1)$$

  4. Synthesis : conclude by if step 2 and 3 are proven then is true for all $n \geqslant 1$

FIBONACCI NUMBERS :EXPLORATION

A method

- think before going to the computations
- try to understand a direction
- gather all the surrounding knowledge (other identities)
- remind some proofs that are similar
- try on small examples
- propose several technical strategies, and choose one
- try to write the different parts of the proof
- rewrite the proof on the draft paper, polish it, optimize it (get rid of unnecessary elements, before writing the proof on the exam sheet

Fibonacci numbers : a typical answer (among many)

Step 1 : the property

Let $n \geqslant 1$

$$\mathcal{P}(n) : \quad F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1}$$

or

$$\mathcal{P}(n) : \quad F(n-1).F(n+1) - F(n)^2 = (-1)^{n+1} \text{seems to me easier to prove} \qquad (3)$$

Step 2 : the base case

Let $n = 1$, we have $F(0) = F(1) = 1$ and $F(2) = F(1) + F(0) = 2$ using the property of Fibonacci's numbers (2). Then by substitution

$$F(0).F(2) - F(1)^2 = 1 \times 2 - 1^2 = 1 = (-1)^{1+1}.$$

and $\mathcal{P}(1)$ is true.

FIBONACCI NUMBERS : A TYPICAL ANSWER (AMONG MANY) (CONT.)

Step 3 : Induction

Let $n \geqslant 1$ and suppose that $\mathcal{P}(n)$ is true ; the goal is to prove $\mathcal{P}(n+1)$ true.
Consider the lefthand size of equation 3 at $n+1$

$$
\begin{aligned}
F(n).F(n+2) - F(n+1)^2 &= F(n)(F(n+1)+F(n)) - F(n+1)(F(n)+F(n-1)) \\
&\quad \text{by splitting one term in each product} \\
&\quad \text{we simplify the expression} \\
&= -(F(n-1)F(n+1) - F(n)^2) \\
&= -(-1)^{n+1} \text{using the induction hypothesis} \\
&= (-1)^{(n+1)+1}
\end{aligned}
$$

consequently if $\mathcal{P}(n)$ is true then $\mathcal{P}(n+1)$ is true.

Step 4 : the conclusion

The property holds for $n = 1$ (base case) the induction step is proven, consequently the property is true for all n

GEOMETRIC SERIES

A summation

Determine the value of the following expression and prove it by two different methods :

$$\sum_{k=0}^{n} \frac{1}{4^k}$$

Method (on your draft)

- **Context of the question** Sum of numbers (series), general term $\frac{1}{4^k}$
- **Aim** : to compute the value, depends on a parameter n
- **compute** the first terms (if necessary, it could help to find a picture)
- **recognize** a general structure or a similar case already seen during the lectures
- **fix** notations,
- **identify** several methods that potentially are adapted to the situation (direct computation, recurrence, system of equations, visual proof,picture,...)
- choose one among the methods and try to explain why it could be fruitful
- try to write the **different** parts of the computation
- **rewrite** the computation on the draft paper, **polish** it, **optimize** it (get rid of unnecessary elements), before writing the proof on the exam sheet

Geometric Series (cont.)

A summation

Determine the value of the following expression and prove it by two different methods :

$$\sum_{k=0}^{n} \frac{1}{4^k}$$

Geometric series

First I note

$$S_n = \sum_{k=0}^{n} \frac{1}{4^k} = \sum_{k=0}^{n} \left(\frac{1}{4}\right)^k$$

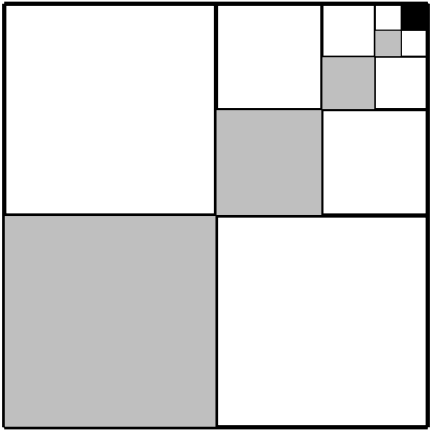
I recognize a geometric series $\sum_{k=0}^{n} a^k$ with ratio $a = \frac{1}{4}$, for $a \neq 1$ we have

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1} - 1}{a - 1}$$

replacing a by $\frac{1}{4}$ we get

$$S_n = \frac{\frac{1}{4^{n+1}} - 1}{\frac{1}{4} - 1} = -\frac{4}{3}\left(\frac{1}{4^{n+1}} - 1\right) = 1 + \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$$

Geometric Series (cont.)



Gray surface$= S_n - 1$, Black surface $= \frac{1}{4^n}$

$$S_n - 1 = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$$

Binomial Coefficients

A Summation

Consider the following expression

$$\sum_{k=0}^{n} k^2 \binom{n}{k} \qquad (4)$$

1. Compute the value of this expression.
2. Propose another formulation of this expression that makes the use of a combinatorial argument easier.

Method (on your draft)

- **Context of the question** Sum of binomial numbers (close to the binomial theorem)
- **Aim** : to compute the value, depends on a parameter n
- **compute** the first terms (if necessary, it could help to find a picture)
- **recognize** a general structure or a similar case already seen during the lectures
- **fix** notations,
- **identify** several methods that potentially are adapted to the situation (direct computation, recurrence, system of equations, visual proof,picture,...)
- choose one among the methods and try to explain why it could be fruitful
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Binomial Coefficients (cont.)

On the draft sheet

Looks like the binomial theorem

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

To get the k in the sum we use derivation

$$n(1+X)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} X^{k-1}$$

to get a product of k we should derive a second time, and we get

$$n(n-1)(1+X)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} X^{k-2} \text{ fixing } X=1 \quad n(n-1)2^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k}$$

This is not exactly what we need we should adapt the result with some algebra

BINOMIAL COEFFICIENTS (CONT.)

Computation

We transform the expression to identify parts of the binomial theorem and derivatives. First recall that (derivation of the binom of Newton two times)

$$\begin{aligned} n(n-1)(1+X)^{n-2} &= \sum_{k=0}^{n} k(k-1)\binom{n}{k}X^{k-2} \\ &= \sum_{k=0}^{n} k^2\binom{n}{k}X^{k-2} - \sum_{k=0}^{n} k\binom{n}{k}X^{k-2} \end{aligned}$$

Then

$$\sum_{k=0}^{n} k^2\binom{n}{k}X^{k-2} = n(n-1)(1+X)^{n-2} + \sum_{k=0}^{n} k\binom{n}{k}X^{k-2}$$

and fixing $X = 1$ we obtain

$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1} = n(n+1)2^{n-2}$$

BINOMIAL COEFFICIENTS (CONT.)

Formulation

The formulation doesn't show clearly the nature of the objects that is counted. It looks like choosing a team Rewrite the expression

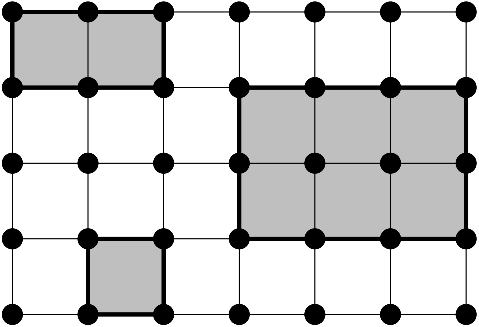
$$\sum_{k=0}^{n} k^2 \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{1} \binom{k}{1}$$

We interpret this expression as the number of teams we can build among n players with an arbitrary size k then we choose a captain and a goalkeeper (the goalkeeper could also be the captain).
By a double counting argument the number of such teams is computed considering to choose first a captain and a goalkeeper, then the rest of the team. Two exclusive cases are possible : either the captain is also the goalkeeper and we have $\binom{n}{1}2^{n-1}$ such teams or the captain and the goalkeeper are different and we have $\binom{n}{1}\binom{n-1}{1}2^{n-2}$ teams. Summing the two expressions we get the final result.

RECTANGLES AND SQUARES IN GRIDS

The problem

Consider a 7×5 grid. In the following figure we have 3 typical rectangles (squares are particular cases of rectangles). The corners of the rectangles are points of the grid and the sides are parallel with the x-axis or the y-axis, a single point of the grid is not considered as a rectangle.



Rectangles and Squares in Grids

Method (on your draft)

- **Context of the question** Enumeration of rectangles : definition of rectangles
- **Aim** : to compute the value, no parameter
- **compute** the first terms (if necessary, it could help to find a picture)
- **recognize** a general structure or a similar case already seen during the lectures
- **fix** notations,
- **identify** several methods that potentially are adapted to the situation (direct computation, recurrence, system of equations, visual proof, picture,...)
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