Combinatorics Basics

Explained by Examples : Subsets, strings, trees

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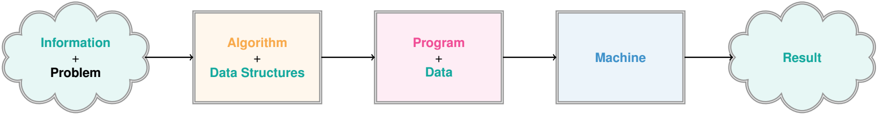
These notes are only the sketch of the lecture : the aim is to apply the basic counting techniques to the binomial coefficients and establish combinatorial equalities.
References : Concrete Mathematics : A Foundation for Computer Science *Ronald L. Graham, Donald E. Knuth and Oren Patashnik* Addison-Wesley 1989 (chapter 5)

COMPUTER SCIENCE



The four faces of a computer science object

Information representation

Encoding, data, numerical information...

Algorithmics

Design, proof, complexity,...

Programming Language

Languages, software engineering...

Architecture

Processor, networks, operating systems...

Reference : **Les quatre concepts de l'informatique**, Didapro, 2011 by Gilles Dowek, INRIA/ENS-Saclay

Abstract Objects in Computer Science

Symbols, Words, and Texts

- 0110111001011101111...
- 270c4fe6205c0f43d3163f566534f308
- grammars, rewriting,...

Trees

- binary trees, binary search trees
- covering trees
- tree structures

Sets

- sets encoding,
- subsets, partitions

Ordering

- permutations
- sequences
- partial orders

Why counting ?

- Characteristics of objects
- Better understanding of the structures
- Counting = Descrtiption Method = Enumeration = Generation

Well Balanced Expressions

The problem

$$((a+(3\times(c+1))-(9+x)\times((5+e)-(4\times 3)))$$ is a well-balanced expression ?

and this one

$$(5+a)\times(2\times 3\times(5/e)))+(4-3\times(e+7))$$

```
CHECK_EXPRESSION(S)
  Data: S sequence of symbols
  Result: True if the expression is well-balanced (else False)
  count = 0
  for i = 0 to length (S) -1
    if S[i] ==' ('
      count = count + 1
    else if S[i] ==')'
      count = count - 1
      if count < 0
        return False
  return count == 0
```

Exercise : Design the algorithm for expressions composed with (), {,} and [,] symbols.

ANALYSIS OF THE PROBLEM

Is the algorithm correct ?

- Formal proof (modify the algorithm to prove it)
- Check on examples (which ones ?) Is it a proof ?
- Enumerate all the possible expressions with $'('$ and $')'$ and check the correctness. Is it a proof ?
- Generate a random set arbitrary large of expressions and check. Is it a proof ?

Aim of the activity :

- Describe the structure, check details, fix notations
- Explore the small cases exhaustively
- Establish algebraic structure, make links with other problems

SUBSET ENUMERATION

$\binom{n}{k}$ is the number of ways to choose k elements among n elements

http://www-history.mcs.st-and.ac.uk/Biographies/Pascal.html

For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{1}$$

Prove the equality by a combinatorial argument
Hint : the number of sequences of k different elements among n is $n(n-1)\cdots(n-k+1)$ *and the number of orderings of a set of size k is* $k!$.

BASIC PROPERTIES

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

Prove it directly from Equation 1

For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Prove it directly from 2
Prove it by a combinatorial argument
Hint : bijection between the set of subsets of size k and ???.

Exercise

Give a combinatorial argument to prove that for all integers $0 \leqslant k \leqslant n$:

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{4}$$

Pascal's triangle

Recurrence Equation

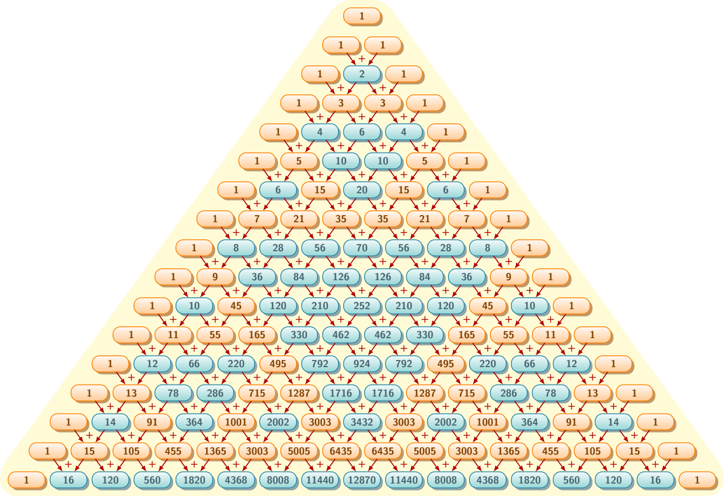
The binomial coefficients satisfy

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{5}$$

Prove it directly from Equation 1
Prove it by a combinatorial argument

Hint : partition in two parts the set of subsets of size k ; those containing a given element and those not.

Pascal's triangle(2)



Thanks to Tikz/Gaborit

THE BINOMIAL THEOREM

For all integer n and a formal parameter X

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k \text{ (Newton 1666)} \tag{6}$$

Prove it by a combinatorial argument *Hint : write* $(1+X)^n = \underbrace{(1+X)(1+X)\cdots(1+X)}_{n \text{ terms}}$ *in each term choose* 1 *or* X*, what is the coefficient of* X^k *in the result (think "vector of n bits").*

Exercises

Use a combinatorial argument to prove :

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Use the binomial theorem to prove (give also a combinatorial argument)

$$\sum_{k=0\ k\ odd}^{n} \binom{n}{k} = \sum_{k=0\ k\ even}^{n} \binom{n}{k} = 2^{n-1}$$

Summations and Decompositions

The Vandermonde Convolution

For all integers m, n, k

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \tag{7}$$

Prove it by a combinatorial argument
Hint : choose k elements in two sets one of size m and the other n.

Exercise

Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{8}$$

Hint : Specify Equation 7

Summations and Decompositions (2)

Upper summation

For all integers $p \leqslant n$

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1} \tag{9}$$

Exercises

Establish the so classical result

$$\sum_{k=1}^{n} \binom{k}{1}$$

Compute

$$\sum_{k=2}^{n} \binom{k}{2} \text{ and deduce the value of } \sum_{k=1}^{n} k^2$$

THE MAIN RULES IN COMBINATORICS (I)

Bijection Rule

Let A and B be two finite sets if there exists a bijection between A and B then

$$|A| = |B| \, .$$

Summation Rule

Let A and B be two **disjoint** finite sets then

$$|A \cup B| = |A| + |B| \, .$$

Moreover if $\{A_1, \cdots A_n\}$ is a partition of A (for all $i \neq j$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=0}^n A_i = A$)

$$|A| = \sum_{i=0}^{n} |A_i| \, .$$

THE MAIN RULES IN COMBINATORICS (II)

Product rule

Let A and B be two finite sets then

$$|A \times B| = |A| \, . \, |B| \, .$$

Inclusion/Exclusion principle

Let $A_1, A_2, \cdots A_n$ be sets

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^k \sum_{S \subset \{1, \cdots, n\}, \, |S|=k} \left| \bigcap_{i \in S} A_i \right| .$$

Exercises

Illustrate these rules by the previous examples, giving the sets on which the rule apply.

THE CENTRAL ROLE OF BIJECTION

Mapping

A **mapping** (function) between X and Y associate to each element x of X a unique element Y

$$\begin{array}{rccc} f: & X & \longrightarrow & Y \\ & x & \longmapsto & y \end{array}$$

f is an **injection** iff

$$\forall (x_1, x_2) \in X^2 \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

f is a **surjection** iff

$$\forall y \in Y \quad \exists x \in X \text{ such that } y = f(x)$$

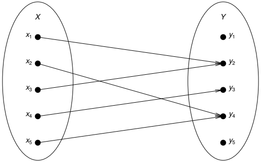
f is a **bijection** iff f is injective **and** surjective

$$\forall y \in Y \quad \exists ! x \in X \text{ such that } y = f(x) \ \ (x \text{ is unique})$$

Mappings and cardinalities

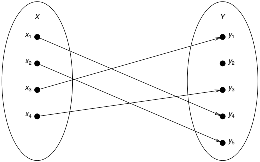
X and Y **FINITE** sets

Typical mapping



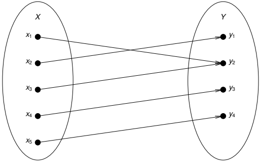
no relation between $|X|$ and $|Y|$

Injective mapping



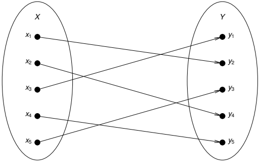
$|X| \leqslant |Y|$

Surjective mapping



$|X| \geqslant |Y|$

Bijective mapping

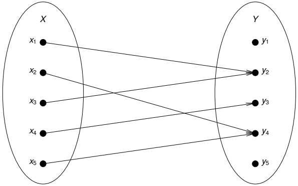


$|X| = |Y|$

What happens when the sets are infinite ?

RECIPROCAL MAPPING

A typical mapping f



Inverse Image

subsets of elements of X (equivalence relation on X)

$$
\begin{aligned}
f^{-1}(y_1) &= \emptyset \\
f^{-1}(y_2) &= \{x_1, x_3\} \\
f^{-1}(y_3) &= \{x_4\} \\
f^{-1}(y_4) &= \{x_2, x_5\} \\
f^{-1}(y_5) &= \emptyset
\end{aligned}
$$

$$f^{-1}(y) = \{x \in X, \text{ such that } f(x) = y\}$$

Combinatorial property :

$$\sum_{y \in Y} \left| f^{-1}(y) \right| = |X|$$

Exercise :
For all the previous combinatorial proofs construct the corresponding functions.

COUNTING FUNCTIONS (EXERCISES)

Let X and Y finite sets

$$\begin{array}{rccc} f: & X & \longrightarrow & Y \\ & x & \longmapsto & y \end{array}$$

- Compute the total number of such functions f
- Compute the number of *injective* functions
- Compute the number of *surjective* functions
- Compute the number of *bijective* functions

Counting relations

Let X be set, a **relation** $\mathcal{R}$ is a part of $X \times X$.
When X is finite, compute the number of relations on X that are

- **reflexive** ($\mathcal{R}$ is reflexive iff $\forall x \in X$ we have $x\mathcal{R}x$)
- **symetric** ($\mathcal{R}$ is symetric iff $\forall (x, y) \in X^2$ we have $x\mathcal{R}y \Longrightarrow y\mathcal{R}x$)
- **antisymetric** ($\mathcal{R}$ is antisymetric iff $\forall (x, y) \in X^2$ we have ($x\mathcal{R}y$ and $y\mathcal{R}x$) $\Longrightarrow x = y$)

$\mathcal{R}$ is **transitive** iff $\forall (x, y, z) \in X^3$ we have ($x\mathcal{R}y$ and $y\mathcal{R}z$) $\Longrightarrow x\mathcal{R}z$
Try to understand why computing the number of transitive relations is hard. OEIS

Distribution problems

Context

Place a set of N objects, called *balls*, into a set of M containers, called *urns*.
Basic situations :

- Labelled balls
- Labelled urns

More constraints :

- at least k balls per urn
- at last k balls per urn
- number of empty urns
- ...

EXAMPLE WITH $N = 3$ AND $M = 2$

	Labelled urns		Unlabelled urns	
labelled balls	urn 1	urn 2		
	123	$\emptyset$		
	12	3	one urn	the other
	13	2	123	$\emptyset$
	23	1	12	3
	1	23	13	2
	2	13	23	1
	3	12		
	$\emptyset$	123		
unlabelled balls	urn 1	urn 2		
	***	$\emptyset$	one urn	the other
	**	*	***	$\emptyset$
	*	**	**	*
	$\emptyset$	***		

Compute the number of configurations in each cell and generalize (if possible).

Derangement

Definition

A derangement of a set S is a bijection on S without fixed point.
Number of derangements of n elements d_n (notation $!n$).

Inclusion/Exclusion principle

$$\begin{aligned} !n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)!, \\ &= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \overset{n\to\infty}{\sim} n! \frac{1}{e}. \end{aligned}$$

Recurrence relation

Show by a combinatorial argument that

$$d_n = (n-1)(d_{n-1} + d_{n-2}) = n d_{n-1} + (-1)^n.$$

Proof of the second equation

First we have the first element Thanks OEIS

n	0	1	2	3	4	5	6	7	8	9	10	...
d_n	1	0	1	2	9	44	265	1854	14833	133496	14684570	...

Suppose that d_n satisfies the recurrence equation $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n \geqslant 2$ with $d_0 = 1$ and $d_1 = 0$.
We will prove by recurrence that $d_n = nd_{n-1} + (-1)^n$ with $d_0 = 1$ (E).

1. base case : this is true for $n = 0$ and $n = 1$
2. Suppose that (E) is satisfied for $n - 1$
   Then $d_{n-1} = (n-1)d_{n-2} + (-1)^{n-1}$, we deduce that $(n-1)d_{n-2} = d_{n-1} - (-1)^{n-1}$.
   Injecting that equality in the recurrence equation of d_n

$$
\begin{aligned}
d_n &= (n-1)(d_{n-1} + d_{n-2}) \\
&= (n-1)d_{n-1} + (n-1)d_{n-2} \\
&= (n-1)d_{n-1} + d_{n-1} - (-1)^{n-1} \\
&= nd_{n-1} + (-1)^n
\end{aligned}
$$

3. The base case and the induction is proven, so is the result

FIBONACCI NUMBERS

Recurrence Equation

$$\begin{cases} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geqslant 2 \end{cases}$$

Interpretation

What kind of situation could be represented by Fibonacci's Numbers ?
Hint : Consider words in $\{0,1\}^n$
Use a combinatorial argument to prove

$$F_n = F_{n-2} + F_{n-3} + \cdots + F_1 + F_0$$

Hint : Consider the last "1"

Imagine other combinatorial equalities

(Undirected) Trees

A tree $\mathcal{T} = (\mathcal{X}, \mathcal{E})$ is an acyclic connected graph

- **connected** : for all $x, y \in \mathcal{X}^2$ there is a path from x to y ($x \rightsquigarrow y$)
- **acyclic** : there are no paths from x to x $x \not\rightsquigarrow x$

Notations

$\mathcal{X}$ set of n nodes
$\mathcal{E}$ set of edges
A **leaf** is a node with exactly one edge and an **internal node** has at least two neighbors.

- Prove that the maximum number of leaves is $n - 1$ and the minimum 2 (for ($n \geqslant 3$).

An undirected graph $\mathcal{T}$ with n nodes is a tree iff

1. $\mathcal{T}$ is acyclic and connected
2. $\mathcal{T}$ is acyclic with a maximal number of edges
3. $\mathcal{T}$ is connected with a minimal number of edges
4. $\mathcal{T}$ is connected with $n - 1$ edges
5. $\mathcal{T}$ is acyclic with $n - 1$ edges
6. for all couple (x, y) of nodes there is a unique path $x \rightsquigarrow y$ joining the two nodes.

Prove the equivalences (with a minimal number of implications).

CAYLEY'S FORMULA

$\mathcal{T}_n$ the set of all trees with n nodes labelled by the first integers $\{1, 2, \cdots, n\}$
T_n the number of such trees.

Phase 1 : small n cases

n	T_n
1	1
2	1
3	3
4	16
5	125
$\cdots$	$\cdots$

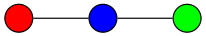
Phase 2 : Intuition of the Formulae

$$T_n = n^{n-2}.$$

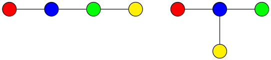
Many proofs (see "Proofs from the Book").
Approach based on an explicit bijection between the set of trees and the a set of words.
Algorithmic as it associates to each tree a unique word with a coding algorithm.
The uniqueness is obtained with a decoding algorithm (H. Prüfer in 1918).

$A_1 = 1$

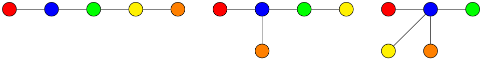
$A_2 = 1 = 2^{2-2}$



$A_3 = \binom{3}{1} = 3 = 3^{3-2}$

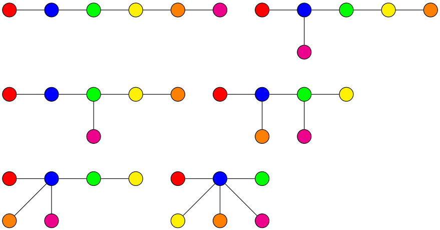


$A_4 = \binom{4}{2}\binom{2}{1} + \binom{4}{1} = 16 = 4^{4-2}$



$$A_5 = \binom{5}{1}\binom{4}{2}\binom{2}{1} + \binom{5}{1}\binom{4}{2}\binom{2}{1} + \binom{5}{1}$$

$$= 60 + 60 + 5 = 125 = 5^{5-2}$$



$$A_6 = \binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{2}{1} + \binom{6}{1}\binom{5}{2}\binom{3}{1}\binom{2}{1}$$

$$+\binom{6}{1}\binom{5}{1}\binom{4}{2}\binom{2}{1} + \binom{6}{2}\binom{4}{2}$$

$$+\binom{6}{1}\binom{5}{2}\binom{2}{1} + \binom{6}{1}$$

$$= 360 + 360 + 360 + 90 + 120 + 6$$

$$= 1296 = 6^{6-2}$$

PRÜFER'S CODING ALGORITHM

Phase 3 : double counting

Find a one to one mapping with another set which cardinality is known.

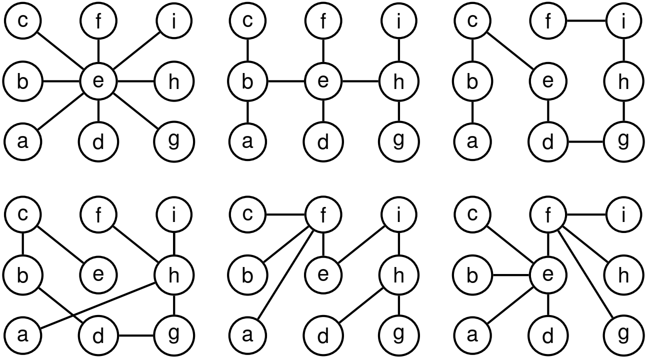
$$\mathcal{T}_n \longleftrightarrow \mathcal{W}_{n-2}$$

$\mathcal{W}_{n-2}$ is the set of words of length $n-2$ over the alphabet $\{1, \cdots, n\}$

CODING (T)
Data: A tree T with labelled nodes (all labels are comparable)
Result: A word of $n-2$ labels

$W \leftarrow \{\}$
for $i = 1$ **to** $n-2$ **do**
  $x \leftarrow$**Select_min** (T)
  `// ` x ` is the leaf with the smallest label`
  $W \leftarrow W+$**Father** (x)
  `// ` **Father** (x) ` is the unique node connected to the leaf ` x
  $T \leftarrow T \setminus \{x\}$ `// remove the leaf ` x ` from tree ` T

LABELLED TREES



PRÜFER'S DECODING ALGORITHM

DECODING (W)
Data: A word $W = w_1 w_2 \cdots w_{n-2}$ of $n-2$ labels in $\{1, \cdots, n\}$
Result: A tree with n nodes labelled from 1 to n

Create n nodes labelled from 1 to n and mark each node by "non selected"
for $i = 1$ **to** $n-2$ **do**
  $x \leftarrow$**Select_min** (W_i)
  `//` x `is the node with the smallest label not in the set` $w_i \cdots w_{n-2}$
  Mark x by "selected"
  Link x and w_i
Link the last two nodes marked "non selected"
return T

Prüfer's Decoding Algorithm

Examples (10 letters words)

0	d	i	g	h	a	c	g	c	f	f
1	e	h	i	e	i	c	a	e	d	d
2	e	f	g	d	g	g	i	b	c	d
3	h	h	g	h	c	f	c	c	d	f
4	i	f	e	c	d	f	a	h	g	f
5	c	b	e	a	g	i	d	i	a	g
6	b	g	g	i	b	b	f	i	b	d
7	e	i	c	c	a	c	f	i	b	d
8	b	i	d	i	e	e	a	g	d	a
9	g	c	b	f	c	f	e	f	b	f
10	b	h	i	a	b	e	b	e	c	h
11	d	e	h	g	f	f	f	b	e	g
12	b	h	i	e	a	d	d	g	h	f
13	g	a	b	h	a	a	g	h	i	i
14	d	h	d	e	i	i	b	f	b	a
15	h	e	c	a	b	a	b	c	h	d
16	i	e	g	i	d	i	e	e	b	g
17	d	g	i	b	e	h	c	e	i	f
18	c	h	a	b	e	f	g	b	h	i
19	h	a	f	b	d	h	c	d	h	g

Questions

- Prove the bijection
- Compute the complexity of coding and decoding
- What kind of data structure could be useful ?
- How degrees are expressed in the coding word ?

Extension : is it possible to build a tree from a list of degrees ?

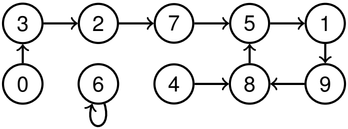
Joyal's Bijection

What set of objects has cardinality n^n ?
Number of mappings from $\mathcal{X}$ on $\mathcal{X}$,
(number of words of size n on an alphabet of size n)

A mapping f

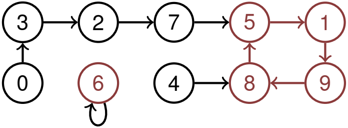
x	0	1	2	3	4	5	6	7	8	9
$f(x)$	3	9	7	2	8	1	6	5	5	8

Graph associated to mapping f



JOYAL'S BIJECTION

Cycles

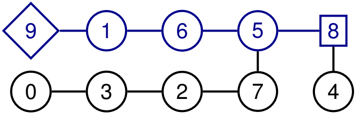


- Each node has an outdegree $= 1$
- Decomposition in **cycles** and **transient** nodes

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	3	9	7	2	8	1	6	5	5	8

- Extract the bijective part
- build a line with the ordered bijective part

Build the tree



- Fix the line between diamond (image of the smallest) and rectangle (image of the greatest)
- Connect the transients and remove arrows

Design the reciprocal algorithm

Generating Function

Newton's Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

One to one correspondance

$$(1+x)^n \longleftrightarrow \binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$$

Generating Function (Power Series)

Sequence $a = \{a_0, a_1, \cdots, a_n, \cdots\}$

$$G_a(x) \stackrel{def}{=} \sum_{n=0}^{+\infty} a_n x^n$$

(formal series, it is not necessary to ensure convergence)

Generating Function (2)

A bijection

Derivation operator

$$
\begin{aligned}
G_a(x) &= \sum_{n=0}^{+\infty} a_n x^n \\
G_a'(x) &= \sum_{n=1}^{+\infty} n.a_n x^{n-1} \\
G_a''(x) &= \sum_{n=2}^{+\infty} n.(n-1)a_n x^{n-2} \\
\cdots & \\
G_a^{(k)}(x) &= \sum_{n=k}^{+\infty} n.(n-1)\cdots(n-k+1)a_n x^{n-k} \\
\cdots &
\end{aligned}
$$

$$G_a(0) = a_0, \ \frac{G_a'(0)}{1!} = a_1, \frac{G_a''(0)}{2!} = a_2, \cdots, \frac{G_a^{(k)}(0)}{k!} = a_k, \cdots$$

Basic Generating Functions

Sequence	$\longleftrightarrow$ Generating function
$1, 1, 1, \cdots, 1, \cdots$	$\frac{1}{1-x}$
$0, 1, 2, 3, \cdots, n, \cdots$	$\frac{x}{(1-x)^2}$
$0, 0, 1, 3, 6, 10, \cdots, \binom{n}{2}, \cdots$	$\frac{x^2}{(1-x)^3}$
$1, c, c^2, \cdots, c^n, \cdots$	$\frac{1}{1-cx}$
$1, 0, 1, 0, \cdots$	$\frac{1}{1-x^2}$
$\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \cdots, \frac{1}{n!}, \cdots$	e^x
$0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$	$\log \frac{1}{1-x}$

GENERATING FUNCTIONS : APPLICATIONS

Order one equation

$$a_n = 1 + n a_{n-1} \quad n \geqslant 1$$

$$G_a(x) - a_0 = \frac{1}{1-x} - 1 + xG'_a(x)$$

Order two equation

$$f_n = f_{n-1} + f_{n-2} \quad n \geqslant 2$$

Counting objects

Number of ways of choosing a dozen doughnuts when five flavors were available.
{chocolate, lemon-filled, sugar, glazed, plain}

$$G(x) = \frac{1}{(1-x)^5} = \cdots$$

Fibonnacci's Numbers

Recurrence equation

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geqslant 2, \quad f_0 \text{ and } f_1 \text{ fixed}$$

$G(x)$ generating function of $\{f_n\}$, $G(x) = \sum_n f_n x^n$

$$G(x) - f_0 - f_1 x = x(G(x) - f_0) + x^2 G(x)$$

Decomposition of he generating function

$$G(x) = \frac{f_0 + (f_1 - f_0)x}{1 - x - x^2} = \frac{A}{1 - \varphi x} + \frac{B}{1 - \overline{\varphi} x}$$

with $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$, $1 - x - x^2 = (1 - \varphi x)(1 - \overline{\varphi} x)$

- Compute A and B and deduce the power expansion of G .
- Use the power series decomposition $\frac{1}{1-cx} = \sum c^n x^n$.
- And deduce a closed formula for f_n

GENERATING FUNCTIONS : ALGEBRA

Sequence $\longleftrightarrow$	Generating function
$a_0, a_1, a_2, \cdots, a_n, \cdots$	$G_a(x)$
$a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots$	$G_a(x) + G_b()x$
$0, a_0, a_1, a_2, \cdots, a_{n+1} \cdots$	$xG_a(x)$
$0.a_0, 1.a_1, 2.a_2, \cdots, na_n \cdots$	$xG'_a(x)$
$a_0b_0, a_0b_1 + a_1b_0, \cdots, a_0b_n + a_1b_{n-1} + ... + a_nb_0 \cdots$	$G_a(x) \times G_b(x)$

PIGEONS AND HOLES

Principle

If you have more pigeons than pigeonholes
Then some hole must have at least **two** pigeons

Generalization

If there are n pigeons and t holes, then there will be at least one hole with at least

$$\left\lceil \frac{n}{t} \right\rceil \text{ pigeons}$$

History

Johann Peter Gustav Lejeune Dirichlet (1805-1859)
Principle of socks and drawers


http://www-history.mcs.st-and.ac.uk/Biographies/Dirichlet.html

Some examples

On integers (from Erdös)

- Every subset A of $\{1, 2, \cdots, 2n\}$ with size $n+1$ contains at least 2 integers prime together
- Every subset A of $\{1, 2, \cdots, 2n\}$ with size $n+1$ contains at least 2 integers a and b such that a divide b

On sequences

Consider a sequence of n integers $\{a_1, \cdots, a_n\}$.
There is a subsequence $\{a_k, \cdots, a_l\}$ such that

$$n \text{ divide } \sum_{i=k}^{l} a_i$$

IRRATIONAL APPROXIMATION

Friends

Let α be a non-rational number and N a positive integer, then there is a rational $\frac{p}{q}$ satisfying

$$1 \leqslant q \leqslant N \text{ and } \left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{qN}$$

Hint : divide $[0, 1[$ in N intervals, and decimal part of $0, \alpha, 2\alpha, \cdots, N\alpha$

Sums and others

1. Choose 10 numbers between 1 and 100 then there exist two disjoint subsets with the same sum.
2. For an integer N, there is a multiple of N which is written with only figures 0 and 1

Geometry

1. In a convex polyhedra there are two faces with the same number of edges
2. Put 5 points inside a equilateral triangle with sides 1. At least two of them are at a distance less than 1
3. For 5 point chosen on a square lattice, there are two point such that the middle is also on the lattice

Graphs

Friends

Six people
Every two are either friends or strangers
Then there must be a set of 3 mutual friends or 3 mutual strangers

Guess the number

Player 1 : pick a number 1 to 1 Million
Player 2 Can ask Yes/No questions
How many questions do I need to be guaranteed to correctly identify the number ?

Sorting

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