Lecture 5 – Maths for Computer Science
Lab. class, More on graphs

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Exercise 1. Vertex-degrees and the existence of paths

LESSON: Experience with graph-theoretic reasoning

Prove the following property.

Property.
If every vertex of graph G has degree $\geq d$, then G contains a *simple* path of length d.

Exercise 2. Graph-theoretic formulation of 2SAT.

Satisfiability problems deal with propositional formulae that are populated by entities that can assume the truth-values TRUE and FALSE.
The entities are *logical variables*.
The *actual* entities that appear in each formula are *logical literals* (instances of logical variables) in either their *true* or *complemented* forms.

- In its *true* form, a literal evaluates to TRUE precisely when its associated variable does.
- In its *complemented* form, a literal evaluates to TRUE precisely when its associated variable evaluates to FALSE.

Exercise 2.

The following expression exemplify the notions.

formula: $\Phi = (\neg x \vee y) \wedge (x \vee \neg y)$

variables: x and y

literals: x and y (true form); $\neg x$ and $\neg y$ (complemented form)

$$
\begin{array}{cccccccccc}
(& \neg x & \vee & y &) & \wedge & (& x & \vee & \neg y &) \\
 & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
 & \text{complemented} & & \text{true} & & & & \text{true} & & \text{complemented} & \\
 & \text{literal} & & \text{literal} & & & & \text{literal} & & \text{literal} &
\end{array}
$$

Exercise 2.

The Satisfiability Problem is specified by a propositional formula Φ that is a *conjunction of disjuncts of logical literals*.

The Satisfiability question is:
Can one assign truth-values to all of the logical variables of formula Φ in such a way that every disjunct evaluates to TRUE?

$$\Phi \quad = \quad C_1 \ \wedge \ C_2 \ \wedge \cdots \wedge \ C_m \tag{1}$$

where:
- each clause $C_i \ = \ \ell_{i,1} \vee \ell_{i,2}$
- Φ has n logical variables

Example.

$$\Phi_1 \;=\; (a \vee \neg b) \;\wedge\; (b \vee \neg c) \;\wedge\; (c \vee \neg a)$$

Exercise 2.

We transform Φ into a directed graph $G(\Phi)$ that has $2n$ vertices and $2m$ arcs.

- For each logical variable x there is one vertex that represents the TRUE literal form of variable x, and a second vertex that represents the FALSE literal form, $\neg x$, of the variable.
- Each clause $C_i = (\ell_{i,1} \vee \ell_{i,2})$ is represented by a pair of arcs.
  - There is an arc $(\neg x_1 \to x_2)$.
  - Symmetrically, there is an arc $(\neg x_2 \to x_1)$.

All paths in $G(\Phi)$ represent logical implications.

Represent the graphs for Φ_1 andΦ_2 .

$$\Phi_1 = (a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a)$$

$$\Phi_2 = (a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a) \wedge (a \vee c) \wedge (\neg a \vee \neg c)$$

Exercise 2.

The idea here is to solve 2SAT by a path problem in graph $G(\Phi)$.

Propositions.

1. If $G(\Phi)$ contains a path from vertex x to vertex y, then it contains a path from vertex $\neg y$ to vertex $\neg x$.
2. If $G(\Phi)$ contains a path from vertex x to vertex y, then for every truth assignment t that satisfies Φ, if t assigns variable x the truth-value $\textsc{true}$, then t also assigns variable y the truth-value $\textsc{true}$.

Deduce an algorithm for solving 2SAT.