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Concepts : Enumeration,
Method : Coding techniques

Cayley's Formula

Consider $\mathcal{T}_n$ the set of all trees with n nodes labelled by the first integers $\{1, 2, \cdots, n\}$ and denote by T_n the number of such trees. The aim of this exercise session is to compute the T_n and prove the Cayley's formula.

There are many proofs of this theorem, some of them are brilliant, references could be found in the book of Aigner & Ziegler (2014) chapter 30. The approach followed in this exercise is based on an explicit bijection between the set of trees and a set of words. The approach is algorithmic as it associates to each tree a unique word with a coding algorithm. The uniqueness is obtained with a decoding algorithm. It has been discovered by H. Prüfer in 1918.

Preliminaries

A tree is an acyclic connected graph (undirected edges), a leaf is a node with exactly one edge.

1. Prove that the maximum number of leaves is $n-1$ and the minimum 2 (for ($n \geqslant 3$).

2. **At home** prove the equivalence between the definitions

   An undirected graph $\mathcal{T}$ with n nodes is a tree iff

   i *$\mathcal{T}$ is acyclic and connected*

   ii *$\mathcal{T}$ is acyclic with a maximal number of edges*

   iii *$\mathcal{T}$ is connected with a minimal number of edges*

   iv *$\mathcal{T}$ is connected with $n-1$ edges*

   v *$\mathcal{T}$ is connected with $n-1$ edges*

   vi *for all couple of nodes there is a unique path joining the two nodes.*

Enumeration with small n

2. For small values of $n = 1, 2, ..., 5$ draw the set $\mathcal{T}_n$. Could you propose a general method for the enumeration ?

3. Make a conjecture on the value of T_n.

Cayley's Formula

A coding algorithm

CODING (T)
Data: A tree T with n labelled nodes (all labels are comparable)
Result: A word with $n-2$ labels

```
W ← {}
for i = 1 to n − 2 do
    x ←Select_min (T) // x leaf with the smallest label
    W ← W+Father (x)
    // Father (x) is the unique node connected to leaf x
    T ← T\{x}// remove the leaf x from tree T
```

Algorithm 1: Prüfer's coding algorithm

4. Run the algorithm on well chosen examples (a star, a line, an ordinary tree).
5. Establish relations between the degree of a node and the number of occurrences of the label in the word.
6. What could be deduced on your conjecture ?

A decoding algorithm

DECODING (W);
Data: A word $W = w_1 w_2 \cdots w_{n-2}$ of $n-2$ labels in $\{1, \cdots, n\}$
Result: A tree with n nodes labelled from 1 to n

```
Create n nodes labelled from 1 to n and mark each node by "non selected";
for i = 1 to n − 2 do
    x ←Select_min (T);
    // x is the node with the smallest label not in the
       set w_i ··· w_{n−2} and not already selected
    Mark x by "selected";
    Link x and w_i;
Link the last two nodes marked "non selected";
return T
```

Algorithm 2: Prüfer's decoding algorithm

7. Run this algorithm on typical words and particular situations.
8. Prove that these algorithms represent bijections between two sets that are reciprocal. That is **DECODING** (**CODING** (T))=T and **CODING** (**DECODING** (W))=W.
9. What could be deduced now on your conjecture ?

References

Aigner, M. & Ziegler, G. M. (2014), *Proofs from THE BOOK*, Springer.