Fundamental Computer Science

Giorgio Lucarelli

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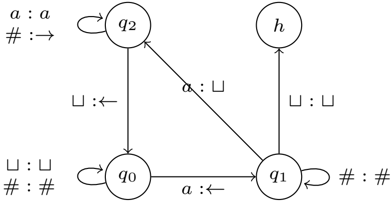
Slides & exercises

`http://moais.imag.fr/membres/giorgio.lucarelli/FCS`

Exercise

$$\Sigma = \{a\}, \quad \Gamma = \{a, \#, \sqcup\}, \quad s = q_0, \quad H = \{h\}$$

$$(q_0, \# \sqcup a^n \underline{a})$$



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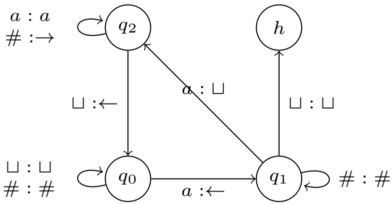

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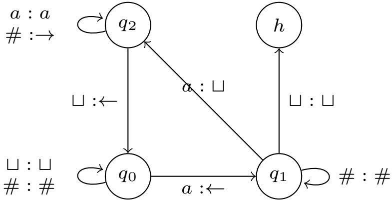

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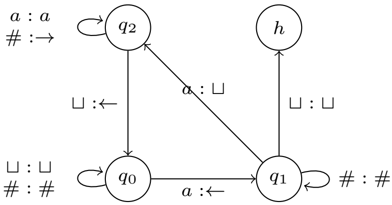

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Definitions

Consider a Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$ such that $H = \{y, n\}$.

Any halting configuration whose state component is y (for "*yes*") is called an **accepting configuration**, while a halting configuration whose state component is n (for "*no*") is called a **rejecting configuration**.

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We say that M **recognizes** (or **semidecides**) a language $L \subseteq \Sigma^*$ if for any string $w \in \Sigma^*$: $w \in L$ if and only if M accepts w.

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A language L is called **Turing-recognizable** (or **recursively enumerable**) if there is a Turing Machine that recognizes it.

Basic theorems

Theorem

If a language L is decidable, then it is Turing-recognizable.

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$$\delta'(q,a) = \begin{cases} n & \text{if } \delta(q,a) = y \\ y & \text{if } \delta(q,a) = n \\ \delta(q,a) & \text{otherwise} \end{cases}$$

$\square$

More definitions

Consider a Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, \{h\})$ and a string $w \in \Sigma^*$. Suppose that M halts on input w and for some $y \in \Sigma^*$ we have

$$(s, \underline{\sqcup} w) \vdash_M^* (h, \underline{\sqcup} y)$$

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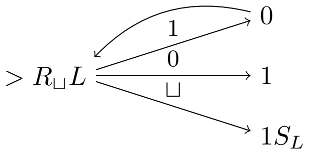

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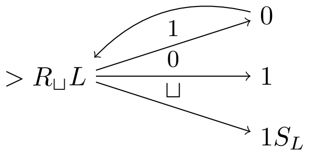

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Computes the function $succ(n) = n + 1$ in binary

Exercise

Prove that the language $L = \{a^n b^n c^n : n \geq 0\}$ is decidable.

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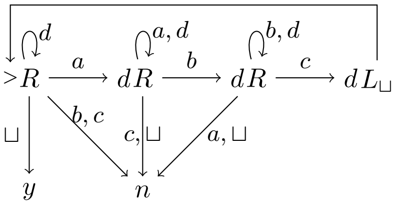

More exercises

- Present Turing Machines that decide the following languages over $\{a, b\}$:
  (a) $\emptyset$
  (b) $\{\epsilon\}$
  (c) $\{a\}$
  (d) $\{a\}^*$

- Give a Turing Machine that *recognizes* the language a^*ba^*b.

Extensions of the Turing Machine

We have already seen an extension:

- write in the tape and move left or right at the same time
- modify the definition of the transition function

  initial: from $(K \setminus H) \times \Gamma$ to $K \times (\Gamma \cup \{\leftarrow, \rightarrow\})$

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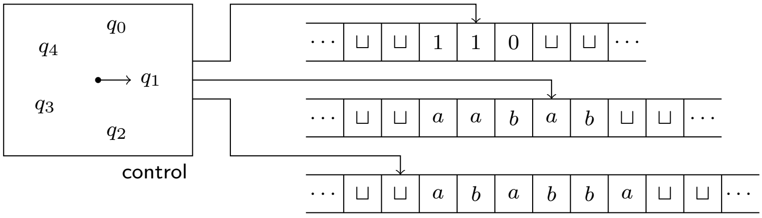
- if the extended Turing Machine halts on input w after t steps, then the initial Turing Machine halts on input w after at most $2t$ steps

Multiple tapes

A k-tape Turing Machine (M) is a sextuple $(K, \Sigma, \Gamma, \delta, s, H)$, where K, Σ, Γ, s and H are as in the definition of the ordinary Turing Machine, and δ is a transition function

$$\text{from} \quad (K \setminus H) \times \Gamma^k \quad \text{to} \quad K \times (\Gamma \cup \{\leftarrow, \rightarrow\})^k$$



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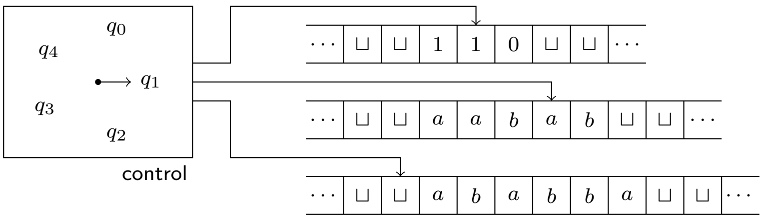

Multiple tapes

Theorem

Every k-tape, $k > 1$, Turing Machine $M = (K, \Sigma, \Gamma, \delta, s, H)$ has an equivalent single tape Turing Machine $M' = (K', \Sigma', \Gamma', \delta', s', H')$.

If M halts on input $w \in \Sigma^*$ after t steps, then M' halts on input w after $O(t(|w| + t))$ steps.

Sketch of the proof:

- M' simulates M in a single tape
- $\#$ is used as delimiter to separate the contents of different tapes
- dotted symbols are used to indicate the actual position of the head of each tape
  - for each symbol $\sigma \in \Gamma$, add both σ and $\dot{\sigma}$ in Γ'
- use the same set of halting states

Multiple tapes

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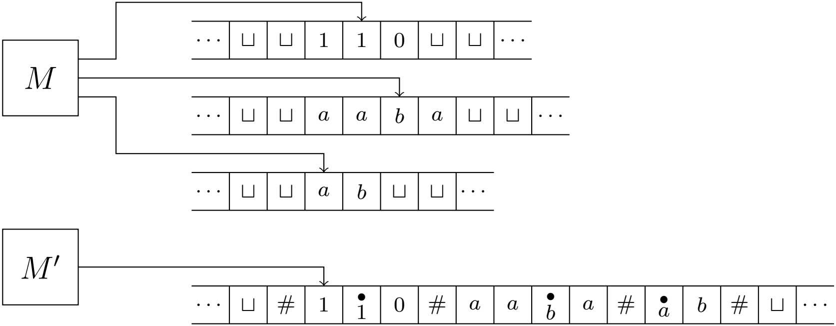

Multiple tapes

Sketch of the proof:

$M' =$ "On input $w = w_1 w_2 \ldots w_n$:

1. Format the tape to represent the k tapes:
$$\# \dot{w}_1 w_2 \ldots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \ldots \#$$
2. For each step that M performs, scan the tape from left to right to determine the symbols under the virtual heads. Then, do a second scan to update the tapes according to the transition function of M.

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Number of steps for M':

1. $O(|w|)$

2. & 3. $O(|w| + t)$ per step $\Rightarrow O(t(|w| + t))$ in total
- size of the tape no more than $O(|w| + t)$

Multiple tapes: conclusion

The multiple tape Turing Machine is <u>**not**</u> more powerful !!

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... and it can be used to simulate functions in an easier way
(a function can use one or more not used tapes)

Multiple tapes: example with $k = 2$ tapes

$$\begin{tikzcd}
> R^{1,2} \arrow[r, "\sigma^1 \neq \sqcup"] \arrow[rr, bend right, "\sqcup^1"'] & \sigma^2 \arrow[l, bend right] & L^2_{\sqcup}\ R^{1,2} \arrow[r, "\sigma^2 \neq \sqcup"] & \sigma^1 \arrow[l, bend right]
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  - $R^{1,2}$: move the head of both tapes on the right
  - σ^2 (as a state): write in the tape 2 the symbol σ
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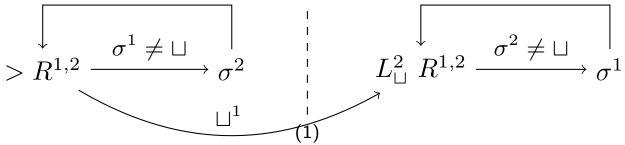

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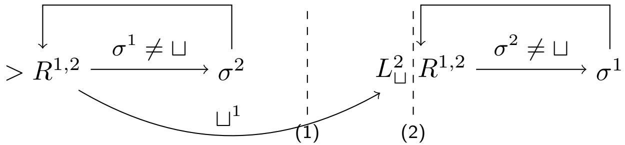

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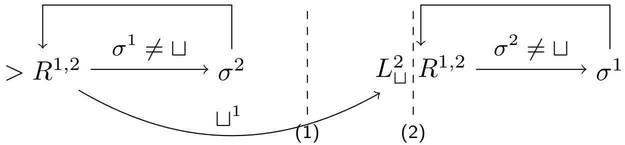

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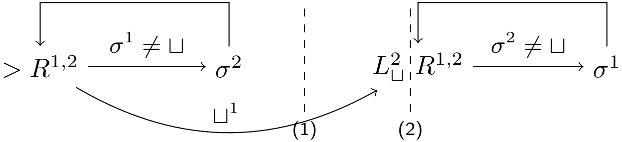

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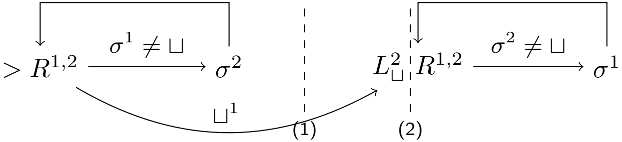

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transforms w to $w \sqcup w$

Multiple tapes: exercise

- Construct a Turing Machine that **adds** two binary numbers.
  Tip: use 2 tapes.

Multiple heads

Definition (informal)

- at each step all heads can read/write/move
- we need a convention if two heads try writing in the same place

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Every multiple head Turing Machine M has an equivalent single head Turing Machine M'.

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Proof (sketch):

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$\cdots$	$\sqcup$	m	y	$\sqcup$	i	n	p	u	t	$\sqcup$	$\cdots$
			$\wedge$								
							$\wedge$				
				$\wedge$							

Multiple heads: example

Give a Machine Turing with two heads that transforms the input $\underline{\sqcup} w$ to $\underline{\sqcup} w \sqcup w$.

- extend notation:
  - $\underline{\sigma}$, $\overline{\sigma}$, $\underline{\overline{\sigma}}$: the position of the 1st, 2nd and both heads, respectively
  - $R^{1,2}$: move both heads on the right
  - σ^2 (as a state): write in the position of head 2 the symbol σ
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$$
\begin{tikzcd}
> R^2_{\sqcup} R^{1,2} \arrow[r, "\sigma^1 \neq \sqcup"] \arrow[d, "\sqcup^1"] & \sigma^2 \arrow[l, bend right] \\
L^{1,2}_{\sqcup} L^2_{\sqcup}
\end{tikzcd}
$$

Two-dimensional tape

Definition (informal)

- move the head left/right/up/down

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Proof (sketch):

- use a multiple tape Turing Machine
- each tape corresponds to one line of the two-dimensional memory

Discussion

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- Observation: a computation in the prototype Turing Machine needs a number of steps which is bounded by a polynomial of the size of the input and of the number steps in any of the extended model

Exercises

- Give an example of a Turing machine with one halting state that does not *compute* a function from strings to strings.
- Give an example of a Turing machine with two halting states, y and n, that does not *decide* a language.
- Can you give an example of a Turing machine with one halting state that does not *recognize* a language?
- Give a Turing Machine which takes as input an integer written in unary and *computes* the binary representation of the same integer (for example $< 11111111111 >_1$ becomes $< 1011 >_2$).

A small exam...

- Construct a Turing Machine that **multiplies** two binary numbers. Tip: use 3 tapes and the machine that performs the addition of two binary numbers as a subroutine.

Instructions

- groups of at most 4 (and at least 3)
- 1 answer per group
- 1 author clearly defined per group
- do not forget to give the names of all members of the group
- you have 30 minutes