UNIVERSITÉ Grenoble Alpes

Master of Science in Informatics at Grenoble

Grenoble INP Ensimag

UE Mathematics for Computer Science

First session exam December 09, 2015 (3 hours)
Only personal hand-written notes are allowed.
Use separated sheets for Part I and Part II.
All problems are independent from each other.
Number of points given for each problem is given for information purposes only and is subject to modifications without notice.

Part I

Exercises

Problem : Perfect Numbers

Some definitions.

A perfect number (*PN* in short) is a number which is equal to the sum of its proper divisors. For instance, the first perfect numbers are the following:

- 6, which has 3 proper divisors, namely, 1, 2 and 3.
  $1 + 2 + 3 = 6$.
- 28 whose 5 proper divisors are: 1, 2, 4, 7 and 14.
  $1 + 2 + 4 + 7 + 14 = 28$.
- 496 has 9 proper divisors, namely, $1, 2, 4, 8, 16, 31, 62, 124$ and 248.
  $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 = 496$.

A Mersenne number is a prime which has the following expression: $2^\alpha - 1$ for some given integer α.

For instance, $3 = 2^2 - 1$, $7 = 2^3 - 1$, $31 = 2^5 - 1$ are primes while $15 = 2^4 - 1$ is not prime... Obviously, there exist primes which are not Mersenne's numbers (like $5, 13$ etc.).

Question 1.1 :

Show that $2^\alpha - 1$ is only prime if α is prime.
Give an example when α is prime and $2^\alpha - 1$ is not.

The objective of the next section is to study some characterizations of perfect numbers. It remains several open questions like the existence of odd perfect numbers or if there are an infinite number of such numbers.

Question 1.2 : Some properties about even *PN*

α is a prime. Let denote by PN_α the number obtained by the following expression: $2^{\alpha-1}(2^\alpha - 1)$ where $2^\alpha - 1$ is a prime.
Show that PN_α is a perfect number and all the perfect numbers have this form.

Question 1.3 : Last digit

Show that the last digit of any perfect number (in usual decimal notation) is 6 or 8.

Question 1.4 : Binary representation

Give the binary representation of PN_3 and PN_5.
Deduce the binary representation of PN_α.

Question 1.5 : Link with triangular numbers Δ_n.

It is easy to remark that $6 = \Delta_3$, verify that 28 is a triangular number.
Show more generally that $PN_\alpha = \Delta_{2^\alpha - 1}$.

Part II

Exercises

Question 2.1 : Integer partition

Compute the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20, \text{ with } x_1, x_2, x_3, x_4 \text{ non-negative integers;}$$

for each of the constraints (all computations should be clearly justified).:

1. no constraint,
2. $x_1 \geqslant 2$ and $x_2 \geqslant 2$;
3. $x_1 + x_2 = 5$;
4. $x_1 \leqslant 3$ and $x_2 \leqslant 3$;
5. $x_1 \leqslant x_2$.

Explain how generating functions could be useful for such computations.

Question 2.2 : Density of numbers

Find the number of 3 digit integers that are not divisible by 4, 5, or 6.

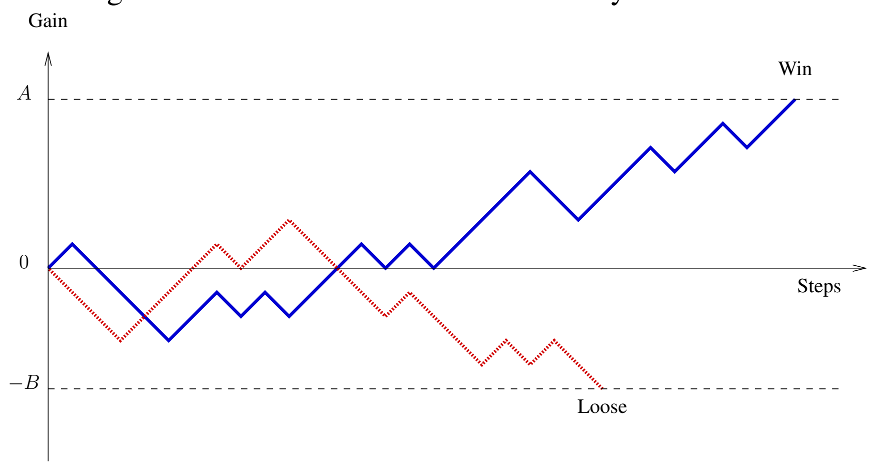
Question 2.3 : Non-decreasing functions

Compute the number of non-decreasing functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, k\}$.

Problem : Gambler's ruin

Consider a simple game with a simple coin, a gambler flip a fair coin if the result is head he wins $\$1$ if the result is tail he looses $\$1$. The game is repeated until he looses his initial fortune B or he wins a fixed amount A.

The evolution of the gambler's fortune could be visualized by



Question 3.1 : Modeling

1. Propose a Markovian model of this situation. Detail the assumptions and establish the properties of the Markov chain.
2. Compute that the probability that gambler is ruined, that is he looses his fortune B before reaching A.
3. Suppose for this question that the coin is unfair and denote by $p \neq \frac{1}{2}$ the probability to get head. Propose a modification of the model. How the ruin probability is modified ?

Consider now the system without any constraints $A = +\infty$ and $B = +\infty$, the evolution on a period of n steps is just a sequence of $+1$ or -1.

Question 3.2 : Number of paths

1. Compute the number of paths leading to a gain of k after n steps.

Denote by C_n the number of paths with $2n$ steps, always having a positive gain and finishing in 0.

2. Compute the values of C_1, C_2, C_3, and C_4, [1].
3. Use a combinatorial argument, prove that

$$C_n = \sum_{i=0}^{n-1} C_{i-1} C_{n-i+1}, \text{ and verify that } C_n = \frac{1}{n+1}\binom{2n}{n};$$

with the convention $C_0 = 1$.

[1] for hackers compute C_5