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Concepts : Enumeration,
Method : Coding techniques

Combinatorial Arguments (do it at home)

Choosing a team You want to choose a team of m people from a pool of n people for your startup company, and from thesempeople you want to choose k to be the team managers. You took the *Mathematics for Computer Science course*, so you know you can do this in $\binom{n}{m}\binom{m}{k}$ ways. But your manager, who went to Harvard Business School, comes up with the formula $\binom{n}{k}\binom{n-k}{m-k}$. Before doing the reasonable thing, dump on your manager, you decide to check his answer against yours.

Start by giving an algebraic proof that your manager's formula agrees with yours. Now give a combinatorial argument proving this same fact.

A curious decomposition Now try the following, more interesting theorem: $n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$

Start with a combinatorial argument. Hint: let $\mathcal{S}$ be the set of all sequences in $\{0, 1, \star\}^n$ containing exactly one $\star$. How would you prove it algebraically?

Covering Let $\mathcal{E}$ a set of n elements. A 2-covering is a couple subsets (A, B) of $\mathcal{E}$ such that $A \cup B = \mathcal{E}$. Compute the number of 2-covering.

Cayley's Formula (mandatory)

Consider $\mathcal{T}_n$ the set of all trees with n nodes labelled by the first integers $\{1, 2, \cdots, n\}$ and denote by T_n the number of such trees. The aim of this exercise session is to compute the T_n and prove the Cayley's formula.

There are many proofs of this theorem, some of them are brilliant, references could be found in the book of Aigner & Ziegler (2014) chapter 30. The approach followed in this exercise is based on an explicit bijection between the set of trees and a set of words. The approach is algorithmic as it associates to each tree a unique word with a coding algorithm. The uniqueness is obtained with a decoding algorithm. It has been discovered by H. Prüfer in 1918.

Preliminaries

A tree is an acyclic connected graph (undirected edges), a leaf is a node with exactly one edge.

1. Prove that the maximum number of leaves is $n - 1$ and the minimum 2 (for ($n \geqslant 3$).

Enumeration with small n

2. For small values of $n = 1, 2, ..., 5$ draw the set $\mathcal{T}_n$. Could you propose a general method for the enumeration ?

3. Make a conjecture on the value of T_n.

Cayley's Formula

A coding algorithm

```
CODING (T)
    Data: A tree T with n labelled nodes (all labels are comparable)
    Result: A word with n − 2 labels
    W ← {}
    for i = 1 to n − 2 do
        x ←Select_min (T) // x is the leaf with the smallest
            label
        W ← W+Father (x)
            // Father (x) is the unique node connected to the
        leaf x
        T ← T\{x}// remove the leaf x from tree T
```

Algorithm 1: Prüfer's coding algorithm

4. Run the algorithm on well chosen examples (a star, a line, an ordinary tree).
5. Establish relations between the degree of a node and the number of occurrences of the label in the word.
6. What could be deduced on your conjecture ?

A decoding algorithm

```
DECODING (W);
    Data: A word W = w_1 w_2 ··· w_{n−2} of n − 2 labels in {1, ··· , n}
    Result: A tree with n nodes labelled from 1 to n
    Create n nodes labelled from 1 to n and mark each node by "non selected";
    for i = 1 to n − 2 do
        x ←Select_min (T);
        // x is the node with the smallest label not in the
            set w_i ··· w_{n−2} and not already selected
        Mark x by "selected";
        Link x and w_i;
    Link the last two nodes marked "non selected";
    return T
```

Algorithm 2: Prüfer's decoding algorithm

7. Run this algorithm on typical words and particular situations.
8. Prove that these algorithms represent bijections between two sets that are reciprocal. That is **DECODING** (**CODING** (T))=T and **CODING** (**DECODING** (W))=W.
9. What could be deduced now on your conjecture ?

References

Aigner, M. & Ziegler, G. M. (2014), *Proofs from THE BOOK*, Springer.