Maths for Computer Science
Divisibility and prime numbers

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Lecture notes MoSIG1

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Motivation

A warm-up question

Every even integer can be written as a sum of two primes.

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- Number Theory is a field where problems are very easy to formalize and understand, but very hard to prove!
- The underlying techniques of number theory are very important in many problems, including cryptography, analysis of algorithms, etc..

The objective of this lecture is to investigate classical results and some related properties.

Some well-known examples

- Goldbach conjecture
  see previous slide
- Perfect numbers conjecture:
  is there any odd perfect number?[1]
- Twin primes:
  are there an infinite number of primes in the form $(n, n+2)$?
- What Fibonacci numbers are prime?
- etc.

[1]Perfect numbers are those which are equal to the sum of their decomposition factors (for instance $28 = 1 + 2 + 4 + 7 + 14$).

Basic results and definitions

Definition.
Let a and b be two integers.
a divides b if there is an integer k such that $ak = b$.

We also say that b is a multiple of a (notice here that a may be negative).

Properties

The following properties are straightforward by applying directly the definition:

1. If $a \mid b$ then $a \mid bc$ $\forall$ integers c
2. If $a \mid b$ and $b \mid c$ then $a \mid c$
3. If $a \mid b$ and $a \mid b + c$ then $a \mid c$
4. $\forall c \neq 0$, $a \mid b$ *iff* $ac \mid bc$
5. If $a \mid b$ and $a \mid c$ then $a \mid sb + tc$ $\forall$ integers s and t

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Proving the last one:

there exist k_1 and k_2 such that $a.k_1 = b$ and $a.k_2 = c$, which implies $a(k_1.s + k_2.t) = sb + tc$ for any s and t.

Greatest Common Divisor

Definition.
$GCD(a, b)$ is the largest number that is a divisor of both a and b.

Proof

Proposition:

$GCD(a, b)$ is equal to the smallest positive linear combination of a and b.[2]

[2] This result is also known as Bezout identity.

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- The proof considers m as the smallest positive linear combination of a and b.
- We prove respectively that $m \geq GCD(a, b)$ and $m \leq GCD(a, b)$ by using the general previous properties.

[2] This result is also known as Bezout identity.

$m \geq GCD(a, b)$

By definition, $GCD(a, b) \mid a$ and $GCD(a, b) \mid b$,
then, $GCD(a, b) \mid sa + tb$ for any s and t (and thus, in particular for the smallest combination).

Then, $GCD(a, b)$ divides m
$m \geq GCD(a, b)$.

$m \leq GCD(a, b)$

We show first that $m \mid a$

First, remark that $m \leq a$
since $a = 1.a + 0.b$ is a particular linear combination.

- There exists a –unique– decomposition $a = q.m + r$ (where $0 \leq r < m$)[3].

[3] The proof is by the division theorem whose proof will be detailed later

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- Recall also that $m = sa + tb$ for some s and t.
- Thus, r can be written as $a - q.m = (1 - qs)a + (-qt)b$ which is a linear combination of a and b.

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- Thus, r can be written as $a - q.m = (1 - qs)a + (-qt)b$ which is a linear combination of a and b.
- However, as m is the smallest one and $r < m$ we get $r = 0$ by contradiction.
- Thus, $a = q.m$

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Symmetrically m divides also b.

Then, $m \leq GCD(a, b)$.

As we have $m \geq GCD(a, b)$ and $m \leq GCD(a, b)$
thus, $m = GCD(a, b)$

An important result

Corollary.

Every linear combination of a and b is a multiple of $GCD(a, b)$

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Properties of the GCD

- Every common divisor of a and b divides GCD(a,b)
- GCD(ak,bk)=k.GCD(a,b) for all $k > 0$
- if GCD(a,b)=1 and GCD(a,c)=1 then GCD(a,bc)=1
- if $a \mid bc$ and GCD(a,b)=1 then $a \mid c$
- GCD(a,b) = GCD(b,rem(a,b))

Euclid's Algorithm

Proposition.
GCD(a,b) = GCD(b,rem(a,b))

rem denotes the reminder of the euclidian division of *a* by *b*.

The property is useful for quickly – iteratively – compute the *GCD* of two numbers.

Geometric interpretation

- Could you provide a graphical proof?

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Attention!
A picture is just an insight,
this is NOT a *proof*.

Formal proof

The idea is to show that the set of common divisors of a and b (called D) is equal to the set of the common divisors of b and $rem(a, b)$ (called D').

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- If $d \in D$, $d \mid a$ and $d \mid b$.
  As $a = q.b + rem(a, b)$ for some q,
  we have $d \mid rem(a, b)$.
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  As $a = q.b + rem(a, b)$ for some q,
  we have $d \mid rem(a, b)$.
  Then, $d \in D'$.
- If $d' \in D'$, $d' \mid b$ and $d' \mid rem(a, b)$.
  d' divides any linear combination of them, in particular
  $q.b + 1.rem(a, b)$
  thus, $d' \mid a$ which proves that $d' \in D$.

The division theorem

Divisibility is not always perfect.

- If one number does not evenly divide another, there is a remainder left.

Division theorem.
Let a and b be two integers such that $b > 0$, then there exists a unique pair of integers q and r such that $a = qb + r$ and $0 \leq r < b$.

Proving this theorem is two-fold: first the **existence**, and then the **uniqueness**.

Existence

Let E be the set of all positive integers in the form $n = a - bz$.
E is not empty (for instance, it contains a) and $E \subset N$, thus, it has a smallest element, say r.
Proving $r < b$ is easy by remarking that $r = a - bz$ for some z and $r - b$ does not belong to E because r is the smallest one.
Thus, $r - b < 0$.

Uniqueness

Let us prove this part by contradiction[4].

- Suppose there exist two such pairs of integers:
  $a = q_i.b + r_i$ for $i = 1, 2$.
- Then, $(q_1 - q_2).b + r_1 - r_2 = 0$, assume $r_1 \geq r_2$
- Thus, b divides $r_1 - r_2$ and $0 \leq r_1 < b$ and $0 \leq r_2 < b$
  $r_1 - r_2 = 0$
- thus, $q_1 = q_2$.

[4] that is common for this kind of proof

Primes

Definition.

A *prime* is an integer with no positive divisor other than 1 and itself (otherwise, it is said a *composite*).
1 is neither a prime nor a composite.

Fundamental theorem of Arithmetic.

Every positive integer n can be written in an unique way as a product of primes: $n = p_1 p_2 ... p_j$ $(p_1 \leq p_2 \leq ... \leq p_j)$.

We have again two results to prove:

1. every integer can be written as the product of primes
2. this factorization is unique

Proof (existence)

The first part is proven easily by a strong induction:
Let assume that all numbers can be decomposed accordingly up to n and let consider $n+1$.

- If it is prime, the decomposition exists (it is it-self)
- if it is a composite, each factor can be expressed as a product of primes by the induction hypothesis.

Proof (uniqueness)

The uniqueness is obtained by contradiction:
Let us first order the primes of the decomposition by increasing values.

$$P_1 \ < \ P_2 \ < \cdots < \ P_{r-1} \ < \ P_r$$

where $P_i \leq n$.
Assume n has two distinct canonical prime factorizations.

$$\langle a_1, a_2, \ldots, a_r \rangle \quad \text{and} \quad \langle b_1, b_2, \ldots, b_r \rangle$$

such that n is equal to both of the following products of primes P_1, P_2, ..., P_{r-1}, P_r.

$$P_1^{a_1} \cdot P_2^{a_2} \cdot \ \cdots \ \cdot P_{r-1}^{a_{r-1}} \cdot P_r^{a_r} \tag{1}$$

$$P_1^{b_1} \cdot P_2^{b_2} \cdot \ \cdots \ \cdot P_{r-1}^{b_{r-1}} \cdot P_r^{b_r} \tag{2}$$

Let us now cancel the longest common prefix.

Because the two products are, by hypothesis, distinct, at least one of them will not be reduced to 1 by this cancellation. We are, therefore, left with residual products of the forms

$$P_i^{a_i} \cdot X \tag{3}$$

$$P_i^{b_i} \cdot Y \tag{4}$$

- Precisely one of a_i and b_i equals 0. Say, $b_i = 0$ while $a_i \neq 0$.
- Products X and Y are composed only of primes that are strictly bigger than P_i.

We have reached the **point of contradiction**:

On the one hand, P_i *must* divide the product Y, because it divides the product $P_i^{a_i} \cdot X$ which equals Y.

On the other hand, P_i *cannot* divide the product Y, because every prime factor of Y is bigger than P_i (and a prime cannot divide a bigger prime).

Three examples

- Perfect numbers
- Little Fermat theorem
- When Fibonacci meets GCD

Perfect Numbers

- Perfect numbers are those which are equal to the sum of their decomposition factors.
  For instance $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$, 496, ...
  Denote the k^{th} by PN_k

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Theorem (Euclide, Euler)

For every Mersenne-prime $2^p - 1$, the number

$$2^{p-1} \cdot (2^p - 1) \; = \; \binom{2^p}{2} \tag{5}$$

is perfect.

With the aid of the Fundamental Theorem of Arithmetic, let us enumerate the factors of $2^{p-1} \cdot (2^p - 1)$

The list consists of two groups:

1. all powers of 2, from $2^0 = 1, \ldots, 2^{p-1}$
2. all products: $(2^p - 1) \times$(a power of 2 from $2^0 = 1, \ldots, 2^{p-2}$)

For $496 = 2^4 \cdot (2^5 - 1)$, the two groups are:

- 1, 2, 4, 8 and 16
- 31, 62, 124 and 248

A fascinating property

The sum of the inverse of not strict perfect numbers is constant equal to 2.

Example for 28:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$$

- Could you prove it? The proof is easy and left as an exercise.

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Moreover, $PN_\alpha = \Delta_{2^\alpha - 1}$

Fermat's Little Theorem

Aside from its exposing an important and basic property of prime numbers, the theorem provides the basis for a valuable algorithm for testing the primality of integers.

Theorem

Let a be any integer, and let p be any prime.

1. The number $a^p - a$ is divisible by p
2. $a^p \equiv a \bmod p$

Examples : $2^3 - 2 = 3 \times 2$, $3^5 - 3 = 5 \times 48$, $4^7 - 4 = 7 \times 2340$, etc.

Proof (sketches)

We provide two proofs for this fundamental result, each providing a different insight on the result.

- We focus on a fixed prime p and argue by induction on the alphabet size a, that $a^p \equiv a \bmod p$.
- We assume for **induction** that $a^p \equiv a \bmod p$ for all alphabet sizes not exceeding the integer b.

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Invoking the restricted form of the Binomial Theorem, we know that

$$(b+1)^p \;=\; (b^p+1) \;+\; \sum_{i=1}^{p-1} \binom{p}{i} b^{p-i} \tag{6}$$

A simple pictorial argument...

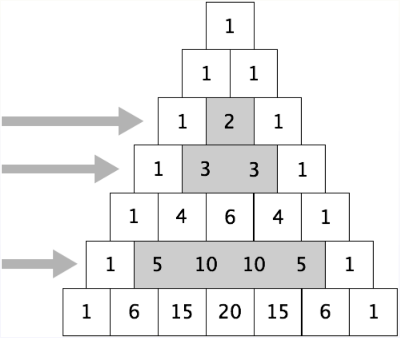


Figure: The "internal" entries of the rows that correspond to prime numbers (in this case, $p = 2$, $p = 3$, and $p = 5$) are divisible by that number.

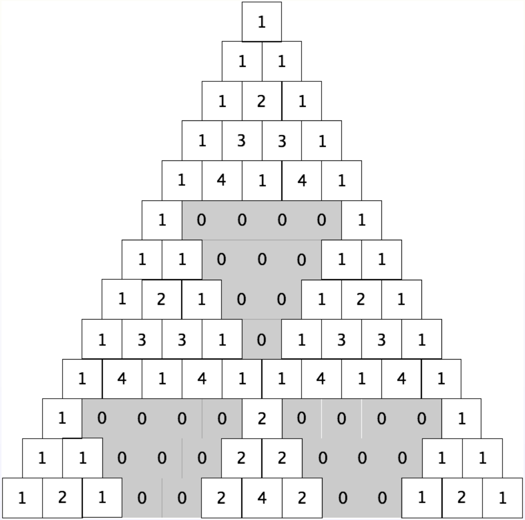


Figure: Pascal's triangle module a prime (5).

A fractal structure

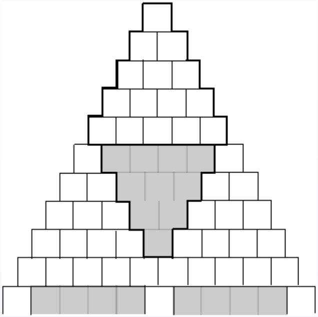
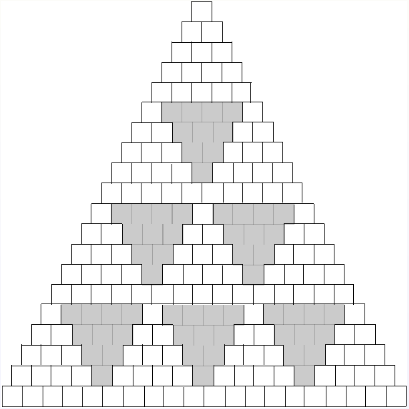


Figure: The shaded area corresponds to 0.



An alternative (combinatorial) proof

a and p be as in the theorem, consider the set of all words/strings of length p over an alphabet/set $\{\alpha_1, \alpha_2, \ldots, \alpha_a\}$ of a symbols.

For instance, when on the binary alphabet $\{0, 1\}$ (so that $a = 2$) and $p = 3$, the set consists of the words:

$$000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111$$

- Except the two groups with the same symbols (000 and 111), there are two groups of size $p = 3$
  namely, the group with one 1 and the group with two 1s

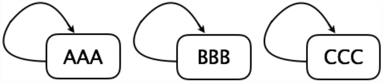


Figure: The 3 strings composed of the same symbol ($a = 3$)

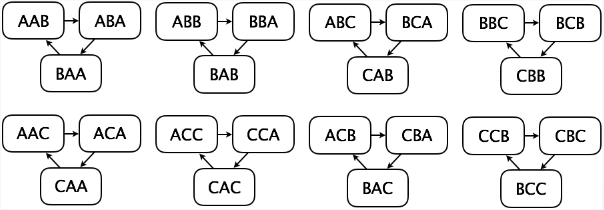


Figure: 8 groups of strings of size $p = 3$ (for $a = 3$).

When divisibility meets Fibonacci numbers...

Let $F(1) = F(2) = 1$ and $F(n+1) = F(n) + F(n-1)$.

Theorem

$GCD(F(n), F(m)) = F(GCD(n, m))$.

Gaining intuition

Recall the Fibonacci progression:
$F(1) = F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5, F(6) = 8, F(7) = 13, F(8) = 21, F(9) = 34, F(10) = 55, \cdots$

Examples:

$GCD(F(6), F(8)) = GCD(8, 21) = 1$
$F(GCD(6, 8)) = F(2) = 1$

$GCD(F(5), F(10)) = GCD(5, 55) = 5$
$F(GCD(5, 10)) = F(5) = 5$

$GCD(F(12), F(18)) = GCD(144, 2584) = 8$
$F(GCD(12, 18)) = F_6 = 8.$