Lecture 6 – Maths for Computer Science
Graphs: Coloring problems

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Lecture notes MoSIG1

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Objective

The purpose of this lecture is to present coloring problems in graphs and analyze their solutions.

Coloring

A *vertex-coloring* of a graph G is an assignment of labels ("*colors*") to G's vertices, in such a way that all of a vertex v's neighbors get different labels than v's.

- The *chromatic number* of a graph G is the smallest number of colors that one can use in crafting a legal vertex-coloring of G.

Content

The notion of graph coloring can be used to computational advantage in a broad variety of situations.

We will study gradually several classes of graphs.

- Graph leveled
- outer-planar graphs
- Planar graphs – The 4-color theorem and Euler's formula.
- Dealing with general graphs

Graph leveled

We start with graphs with small chromatic numbers by focusing on 2-*colorable* graphs.
It is not difficult to characterize these graphs structurally.

Definition
A graph G is *leveled* if there exists an assignment of *level-numbers* $\{1, 2, \ldots, \lambda\}$ to the vertices of G in such a way that every neighbor of a vertex having level-number ℓ has either level-number $\ell + 1$ or level-number $\ell - 1$.

Proposition

A graph G has chromatic number 2 if, and only if, it is leveled.

Proof (sketch)

- Consider first that G is a leveled graph.
  Then, labeling each vertex of G with the (odd-even) parity of its level provides a valid 2-coloring.

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  Then, labeling each vertex of G with the (odd-even) parity of its level provides a valid 2-coloring.
- Consider that G is 2-colorable.
  Pick any vertex v_0 of G and assign it to be the unique vertex on level 1.
  Next, assign all neighbors of v_0 to level 2.
  Continuing iteratively, say that the largest level-number that we have employed (i.e., assigned vertices to) is ℓ.
  Then, we now assign to level $\ell + 1$ all neighbors of level-ℓ vertices that have not yet been assigned to a level.
  Because G is 2-colorable, this process colors every vertex of G.

Moreover, for each vertex v of G:

- Vertex v is assigned a single color.
- If the shortest path from v to vertex v_0 has length ℓ, then v is assigned to level $\ell + 1$.
- Each neighbor of level-$(\ell + 1)$ vertex v is assigned to level ℓ or level $\ell + 2$, as required.

Thus, if we color each vertex v of G by the parity of its assigned level each edge of G connects a vertex of one color with a vertex of the other color.

Applications

We can now show that the following graphs are 2-colorable.

(a) every tree (which includes any path-graph)

(b) every cycle-graph of order n where n is even

(c) every (m, n)-mesh graph

(d) every (m, n)-torus graph with m and n both even

(e) every hypercube H_n

Detail of proof (e)

(e) Each edge of a hypercube H_n connects a vertex $v = \beta_1\beta_2\cdots\beta_n$, where each $\beta_i \in \{0,1\}$, to a vertex $v' = \beta'_1\beta'_2\cdots\beta'_n$ where $\beta_j \neq \beta'_j$ for precisely one j.

Therefore, the following aggregation of vertices of H_n into sets $S_0, S_1, \ldots, S_n$ provides a valid leveling of H_n.

Assign vertex $v = \beta_1\beta_2\cdots\beta_n$ to set S_k precisely if k of the bits β_i equal 1.

The successive levels are associated to the colors alternatively.

outer-planar graphs

Definition

A graph G is *outer-planar* if it can be drawn by *placing its vertices along a circle in such a way that its edges can be drawn as non-crossing chords of the circle*.

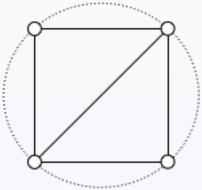
The latter condition is equivalent to demanding that G's edges can be drawn within the circle without any crossings.

- K_3 is outer-planar (straighforward)
- K_4 is not outer-planar
- $K_{2,3}$ is not outer-planar

However, both last graphs are planar...

Proof

K_4 is not outer-planar



It is usually not so easy to prove a negative result.
Can you *prove* this result?

Another example

Proposition

Every Tree is outer-planar.

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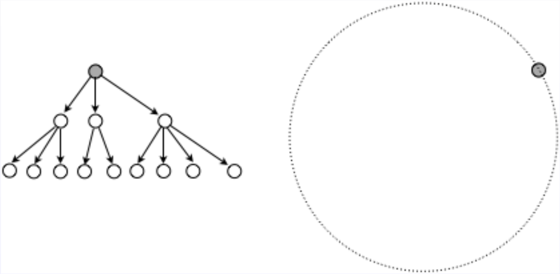

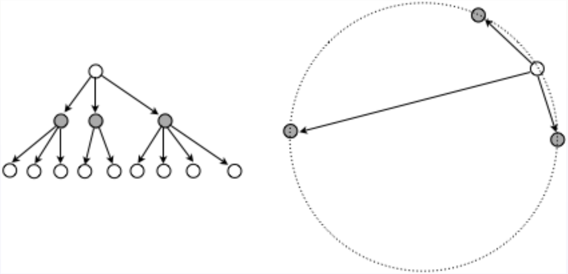

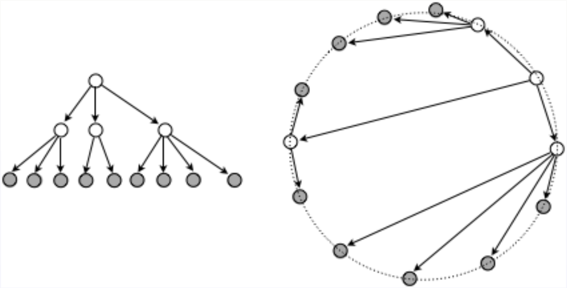

Challenge

Using the intuition gained in the figure on a specific sample, write a formal proof for any tree.

The 3-coloring theorem of outer-planar graphs.

Proposition

Every outer-planar graph is 3-colorable.

A technical Lemma.

Lemma.
Let G be an n-vertex outer-planar graph.

(a) For maximal G: If $n \geq 3$, then at least one of G's vertices has degree 2.
(b) For all outer-planar G: If $n \geq 1$, then at least one of G's vertices has degree ≤ 2.

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(a) For maximal G: If $n \geq 3$, then at least one of G's vertices has degree 2.
(b) For all outer-planar G: If $n \geq 1$, then at least one of G's vertices has degree ≤ 2.

We remark first that part (b) follows immediately from (a) because adding edges to a non-maximal graph — with the goal of making it maximal — can never reduce the degree of any vertex.
Thus, we turn to crafting a proof of part (a).

Proof of Part (a)

We visualize each n-vertex maximal outer-planar graph G in terms of the drawing that witnesses its outer-planarity.
We name G's vertices $0, 1, \ldots, n-1$, in their clockwise order around the circle in the drawing.

Proof of Part (a)

We visualize each n-vertex maximal outer-planar graph G in terms of the drawing that witnesses its outer-planarity.
We name G's vertices 0, 1, . . . , $n - 1$, in their clockwise order around the circle in the drawing.
G's edges come in two groups.

- There are the *ring edges*, i.e., the ones that go around the circle.

  These are edges
  $(0, 1),\ (1, 2),\ (2, 3),\ \ldots,\ (n - 2, n - 1),\ (n - 1, 0)$

- There are the *chordal edges*.

  In the drawing, these are non-crossing chords of the circle.

Study of small graphs

We start by analyzing outer-planar drawings of graphs having three, four, and five vertices, in order to develop intuition.

$n = 3$ *vertices.*
The unique 3-vertex maximal outer-planar graph has three ring edges

$$(0, 1),\ (1, 2),\ (2, 0)$$

and no chordal edges. All three of its vertices have degree 2.

4 vertices

There are two 4-vertex maximal outer-planar graphs.
Both graphs have four ring edges

$$(0,1),\ (1,2),\ (2,3),\ (3,0)$$

and one chordal edge.

The graph with chordal edge $(0,2)$ has two vertices of degree 2, namely, 1 and 3.

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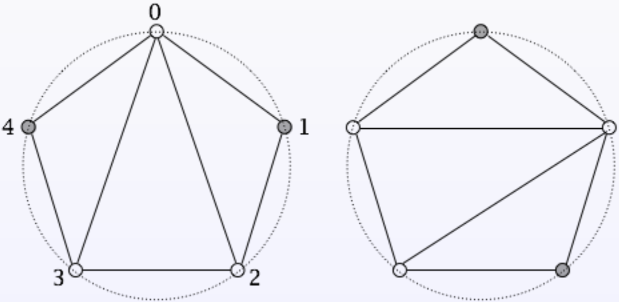
5 vertices

There are five 5-vertex maximal outer-planar graphs.
All graphs have five ring edges and two chordal edges (which share an endpoint).
Let show on two of them:



Large graphs

Extension.

Given any such previous drawing, one cannot add a chordal edge to the drawing without crossing an existing chordal edge.

Let turn this intuition into a formal proof.

Proof of the 3-coloring

The proof is by induction on the number of vertices n.

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Base case. It is very easy to find 3-colorings of outer-planar graphs with ≤ 3 vertices.

Inductive assumption. Assume that every outer-planar graph having $< n$ vertices is 3-colorable.

Inductive extension. Focus on an arbitrary n-vertex outer-planar graph G.

By the previous Lemma, G has a vertex v of degree ≤ 2.
Let us remove it from G, along with its *incident* edges (call the resulting graph G').

- G' is clearly outer-planar.
- G' has less than n vertices.

By our inductive hypothesis, G' is 3-colorable.

Now, we can reattach vertex v to G' by replacing the edges that attach v to G.

Moreover, we can now color v using whichever of the three colors on G is *not* used for v's neighbors in G.

Once we so color v, we will have a 3-coloring of G.

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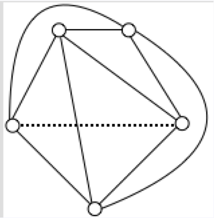
- This *proof mechanism* will be extended for planar graphs.

Planar graphs

Definition.
A graph is planar if it can be drawn without any crossing edges.

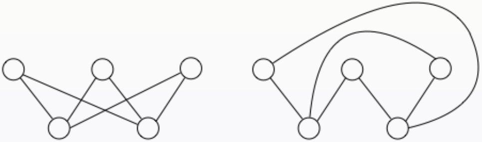
Example

- K_5 is not planar



Other examples

- $K_{2,3}$ is planar



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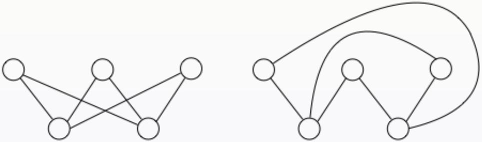

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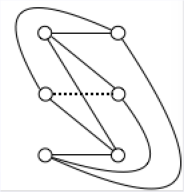


The fundamental theorem

Theorem.
Every planar graph is 4-colorable.

A century-plus attempt to prove that four colors suffice for planar graphs culminated in one of the most fascinating dramas in modern mathematics: Appel and Haken with the help of their computer crafted a proof in 1974.

A flavour of the proof

And, the backstory supplies ample motivation for the proofs, we present the six-color and five-color analogues of the Theorem.

The last one contains the main ideas of the proof.

The Six-Color Theorem for planar graphs

The first step in showing that every planar graph can be vertex-colored using six colors resides in the following analogue for planar graphs of the previous result which asserts that every outer-planar graph has a vertex of degree 2.

Lemma.
Every planar graph has a vertex of degree ≤ 5.

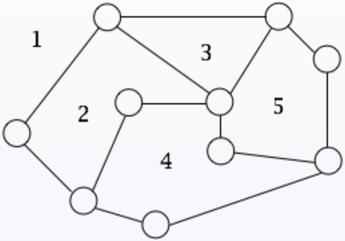
Proof of the Lemma

Let us focus on a planar drawing of a (perforce) planar graph G which has n vertices, e edges, and f *faces*.

Definition

A *face* in a drawing of G is a polygon whose sides are edges of G, whose points are vertices of G, and whose interiors are "empty" (no edge of G crosses through a face).

Example



A planar graph of order 10 with 13 edges and 5 faces.

Euler formula

Let us present an important result that will be proved at the end of this lecture.

Proposition

Let G be a planar graph having n vertices and e edges. For every f-face planar drawing of G, we have an invariant:

$$n - e + f = 2 \tag{1}$$

Back to the proof of the Lemma

Let simplify the setting by assuming that G is *connected* and that it is a *maximal* planar graph[1]

This assumption of maximality only strengthen's the Lemma's conclusion by (apparently) making it more difficult to find a small-degree vertex.

[1] meaning that one cannot add any new edge to the drawing without crossing an existing edge and, thereby, destroying planarity.

n is given, let us enumerate the two other parameters f and e.

Because G is a *maximal* planar graph, we have:

1. Each face of G is a 3-cycle (hence involves three vertices).
2. Each edge of G touches two faces.
3. Each edge of G touches two vertices.

Let us now put these facts together:

$$f = \frac{2}{3}e$$

Assume, for contradiction, that every vertex of G had degree ≥ 6, thus, $e \geq 3n$

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Incorporating these two bounds into Euler's Formula, we arrive at the following contradiction.

$$2 \;=\; n \;-\; e \;+\; f \;\;\leq\;\; \frac{1}{3}e \;-\; e \;+\; \frac{2}{3}e \;\;=\;\; 0$$

This contradiction proves that every planar graph must have a vertex of degree at most 5.

Finishing the proof

This proof is identical to the one 2-color theorem of outer-planar graphs.

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This proof is identical to the one 2-color theorem of outer-planar graphs.

- Base of the induction:
  For outer-planar graphs, "small" means "≤ 3 vertices". For planar graphs, it means "≤ 4 vertices".

- Induction:
  remove from G a vertex v of smallest degree d_v, together with all incident edges

  For outerplanar graphs, we guaranteed that $d_v \leq 2$ and for planar graphs, we guaranteed that $d_v \leq 5$.
- inductively color the vertices of the graph left after the removal of v

  Let us denote by G' the graph obtained by removing v from G. Then:
  For outer-planar graphs, we color G' with ≤ 3 colors, for planar graphs, we use our inductive assumption that G' can be colored with ≤ 6 colors.

- reattach v via its d_v edges and then color v.

  Note that the coloring guarantee in both results allows us to use $d_v + 1$ colors to color G.
  Because v has degree d_v, it can have no more than d_v neighboring vertices in G', so our access to $d_v + 1$ colors guarantees that we can successfully color v.

The proofs of the 3-colorability of outer-planar graphs and the 6-colorability of planar graphs thus differ only in the value of d_v.

The 5-colors theorem

This part is more technical and it is not supposed to be deeply mastered.

We proceed again by induction.

- **Base case.**
  Because the 5-clique K_5 is obviously 5-colorable, so also must be all graphs with ≤ 5 vertices.
  Therefore, we know that any non-5-colorable graph would have ≥ 6 vertices.

Induction

- **Inductive hypothesis.** Assume, for induction, that every planar graph having $\leq n$ vertices is 5-colorable.
- **Inductive extension.**
  By the previous Lemma, the planar graph G has a vertex v of degree ≤ 5.
  The remainder of the proof focuses on the graph G, its minimal-degree vertex v, and on v's ($d_v \leq 5$) neighbors in G.
- Assume that there were a coloring of G's vertices which used no more than 4 colors to color v's neighbors.
  We could, then, produce a 5-coloring of G by using the following analogue of the coloring strategy we used to prove the last 6-color proposition.

- In order to proceed toward a contradiction, we must understand what structural features of G make it impossible to use only four colors on v's neighbors while 5-coloring G. There are three important situations to recognize.

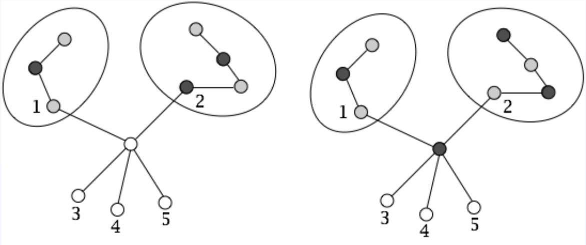
Case 1. Vertex v has degree ≤ 4.
By definition, we need at most four colors to color v's neighbors in this case.
In all remaining cases, vertex v has precisely five neighbors or else, we would have invoked Case 1 to color G with five colors.

Case 2. For some 5-coloring of G, at least two neighbors of v get the same color.

Case 3. All the neighbors get a different color.

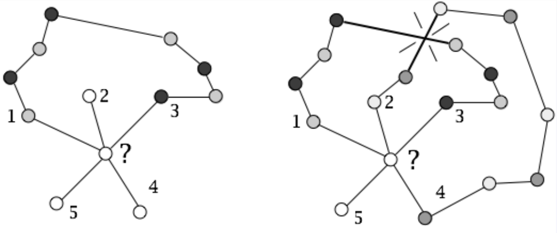
Case 3.a

Two neighbors belong to two connected components (once v is removed).



Case 2.b

All neighbors belong to the same component.



Euler Formula

We propose two proofs for this result.

Validation via structural induction.
This approach validates the formula by growing a planar graph G inductively, edge by edge.

Validation via deconstruction.
Let us be given a planar graph G that has n vertices, e edges, and f faces. We validate the formula by deconstructing G and showing that each step in the process preserves an *invariant*.

First proof

Base case. The Formula clearly holds for the smallest planar graphs, including the smallest interesting one, C_3, which has $n = e = 3$ and $f = 2$ (the inner and outer faces of the "triangle").

Inductive hypothesis.

Assume that the Formula holds for a given graph G.

Inductive extension. We extend our induction by growing the current version of G, by adding a new edge. Two cases arise.

- *The new edge connects existing vertices.* In this case, this augmentation of G increases the number of edges (e) and the number of faces (f) by 1 each, while keeping the number of vertices (n) unchanged.
- *The new edge adds a new vertex, which is appended to a preexisting vertex.* In this case, this augmentation of G keeps the number of faces (f) unchanged while it increases by 1 both the number of edges (e) and the number of vertices (n).

Second proof

Focus on the following two-phase process.

Phase 1. Iterate the process of removing edges from G until some edge-removal reduces G to a graph with a single face.

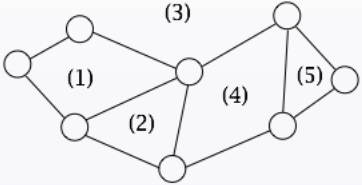
This termination condition is equivalent to stopping when the remaining graph is a (connected) tree.

If the graph remaining at some step contains an edge that is shared by two distinct faces, then remove any such edge.

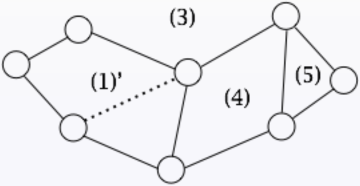
Phase 2. Iterate the following process of removing vertices from the tree produced by Phase 1, until only one vertex remains.

Remove any leaf of the current tree, together with its incident edge.

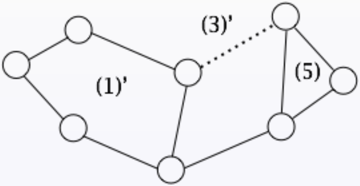
Illustration: the Initial graph



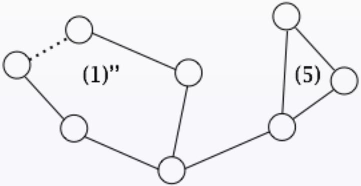
Phase 1 – Step 1



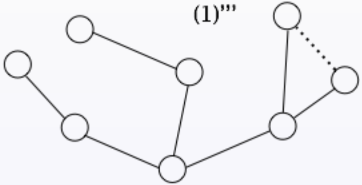
Step 2



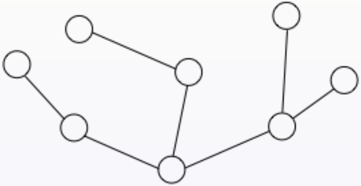
Step 3



Step 4

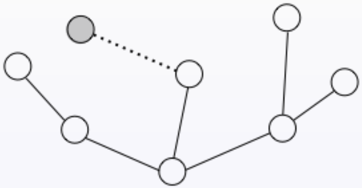


Step 5



We stop when there is only one face left.
The graph is a tree.

We continue in Phase 2 – Step 1



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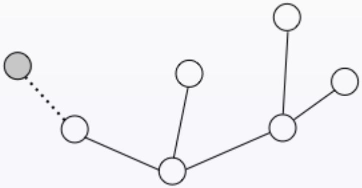

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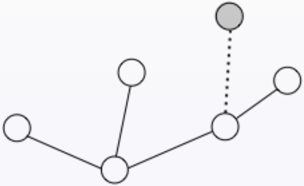

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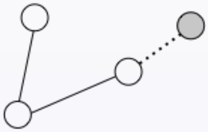
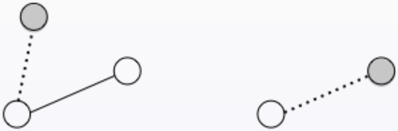
Step 5



Step 6



The process stops when it remains only one vertex.

Analysis

There is an invariant in both phases:

- f is decreasing and e is decreasing.
- both n and e are decreasing.

Thus, ϕ remains constant.

At the end, $f = 1$, $e = 0$ and $n = 1$, $\phi = 2$

Relations between outer-planar and planar graphs

Proposition.

Let G be an outer-planar graph. Then:

(a) G is planar.

(b) Every subgraph of G is outer-planar.

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Let G be an outer-planar graph. Then:

(a) G is planar.
(b) Every subgraph of G is outer-planar.

Proof.

(a) G's planarity can be inferred from our ability to draw G's edges as non-crossing chords of the circle.

(b) We can produce a drawing of any subgraph G' of G that witnesses G''s outer-planarity by erasing some vertices and/or some edges from our outerplanarity-witnessing drawing of G. These erasures cannot introduce any edge-crossings.