UE Mathematics for Computer Science

First session exam December 14, 2020 (3 hours)

Important information.
Read this before anything else!

- ▷ Only one personal hand-written sheet (2 pages) is allowed.
- ▷ Any printed document is not authorized during the exam, excepted dictionaries. Books are not allowed though.
- ▷ Please write your answers to each part on separate sheets of papers (2 separate sheets corresponding to Part A and Part B of the exam).
- ▷ The different exercises are completely independent. You are thus strongly encouraged to start by reading the whole exam. You may answer problems and questions in any order but they have to be written according to the original order on your papers.
- ▷ All answers should be well-argued to be considered correct.
- ▷ All exercises are independent and the total number of points for all problems exceeds 20. You can thus somehow choose the problems for which you have more interest or skills.
- ▷ The number of points alloted to each question gives you an estimation on the expected level of details and on the time you should spend answering.
- ▷ Question during the exam: if you think there is an error in a question or if something is unclear, you should write it on your paper and explain the choice you did to adapt.
- ▷ The quality of your writing and the clarity of your explanations will be taken into account in your final score. **The use of drawings to illustrate your ideas is strongly encouraged** but is not considered as proofs.

Indicative grades

	Part A			Part B		
Exercises	I	II.1	II.2	III.1	III.2	III.3
points	8	2	3	6	4	4

Part A : Proofs, Recurrences, and Graphs

I. Counting triangles

We are interested in this problem in solving a classical combinatorial puzzle.

We consider a series of large triangles, denoted by T_k, that are composed of smaller equilateral triangles (called the basic triangles) whose length of the three sides are equal to 1. The series is built by adding at the bottom of the current large triangle of rank k a row of basic triangles as shown in figure 1. The game consists in the enumeration of all the triangles that compose the large ones (including the basic ones and all the other upper-sized triangles).

In other words, we want to determine how many triangles are contained into T_k for all $k \geqslant 1$. Let us denote this number by N_k and determine the first elements of the progression:

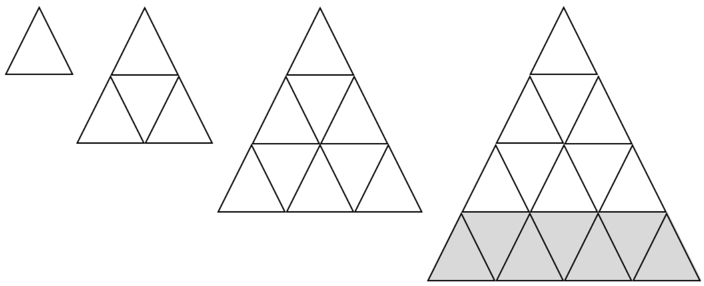

Figure 1: The 4 first terms of the progression of the T_k (from left to right). Each one is built by adding a row of basic triangles on the basis of the previous one (as shown for T_4).

(i) Of course, the first one (T_1) contains only one basic triangle, thus, $N_1 = 1$.

(ii) We have two types of triangles in the second triangle of the progression (T_2), namely a large one whose side is equal to 2 and 4 basic triangles, thus $N_2 = 1 + 4 = 5$.

(iii) Similarly, there are 3 types of triangles contained into T_3: A large one of side 3, 3 medium triangles of side 2 and 9 basic ones, then $N_3 = 1 + 3 + 9 = 13$.

I.1. Determine the number of triangles contained into the fourth term of the progression T_4.

Hint: You can proceed as before by the enumeration of triangles by increasing size i and determine each $N_4^{(i)}$ for $1 \leqslant i \leqslant 4$.

Let us now generalize to any value of k.

I.2. Determine the recurrence equation for computing T_k knowing T_{k-1}, T_{k-2} and T_{k-3}.

We ask here for a detailed and well-argued answer.

Hint: The expression is as follows:
For even k : $N_k = 3(N_{k-1} - N_{k-2}) + N_{k-3} + 2$;
For odd k : $N_k = 3(N_{k-1} - N_{k-2}) + N_{k-3} + 1$;
where $k \geqslant 3$ and $N_0 = 0$, $N_1 = 1$ et $N_2 = 5$.

We turn now to the problem of computing *efficiently* the kth term of this progression.

I.3. Compute the 8 first values of the T_k.

We can compute them directly by the previous expression or using a trick that considers the new derived progression: $\Delta_k = N_{k+1} - N_k$, and then, again by computing the difference $\Gamma_k = \Delta_{k+1} - \Delta_k$ and finally, $\Gamma_{k+1} - \Gamma_k$.

I.4. Draw the corresponding table whose first row is composed by $k = 1, 2, 3, \ldots$, the second row by the Δ_k and the third one by the Γ_k and the fourth one by the differences. Then, derive a low-cost method that allows to fill the table for any k.

This method is much more efficient than the direct computations using the recurrence equations, however, it can even be used further for deriving a close formula (which means: being able to compute directly the value of N_k for any k).

The process is to concentrate to the last row of the previous table and focus on the even k. The values of the previous row (the one with the Γ) can be obtained by simple polynomials of degree 1.

I.5. Determine this polynomial in k.

Hint: we can proceed similarly for the previous row (the one corresponding to the Δ) targeting a polynomial of degree 2 and then again for the N_k with polynomial of degree 3.

I.6. Determine these polynomials for the case of even k.

I.7. Extend the result to odd k.

II. Some exercises

II.1. A little result in coloring graphs

We showed in the class (using a drawing) that a tree is outer-planar.

Prove formally the previous result.

II.2. The density of divisible pairs of numbers

The aim here is to prove the following proposition for every positive integer n.

If you remove *any* $n + 1$ integers from the set $S = \{1, 2, \ldots, 2n\}$, then the set of removed integers contains at least one pair p and $q > p$ such that p divides q.

Hint: Organize the $2n$ numbers of set S into subsets, based on the largest power of 2 that divides them.

II.2.a. Detail the result for $n = 7$ (set $S = \{1, 2, \ldots, 14\}$).

II.2.b. Prove formally the general proposition for every positive integer n.

Part B : Counting and Coin Tossing

III. Distribution problems

In a distribution problem, we are to place a set of objects, called *balls* into a set of containers, called *urns*. Some combinatorial questions occur on the number of possible distributions under some constraints on the characteristics of the sets of balls and the set of urns (number, labelling, capacity,...).

In the problem we not N the number of balls and M the number of urns. If we can distinguish the balls (resp urns) we label them from 1 to N (resp from 1 to M).

III.1. **Classification of some distribution problems**
Consider a first example with $N = 3$ and $M = 2$. The different distributions are given in the following table

	Labelled urns		Unlabelled urns	
labelled balls	urn 1	urn 2	one urn	the other
	123	$\emptyset$	123	$\emptyset$
	12	3	12	3
	13	2	13	2
	23	1	23	1
	1	23		
	2	13		
	3	12		
	$\emptyset$	123		
unlabelled balls	urn 1	urn 2	one urn	the other
	$***$	$\emptyset$	$***$	$\emptyset$
	$**$	$*$	$**$	$*$
	$*$	$**$		
	$\emptyset$	$***$		

III.1.a. Compute the number of distributions for each situation and establish a formula for N balls and $M = 2$ urns. If you cannot get a closed formula give a way to algorithmically compute these numbers.

III.1.b. For N balls and $M = 2$ urns, what is changed if we impose that each urn has at least one ball ?

III.1.c. Compute the number of distributions for each situation and establish a formula for $N = 2$ balls and M urns. If you cannot get a closed formula give a way to algorithmically compute these numbers.

III.1.d. For the general case give a closed formula for labelled urns.

III.1.e. For the general case with unlabelled urns give a recurrence equation satisfied by these numbers and fill a table for $1 \leqslant M, N \leqslant 5$.

III.2. **Probabilistic distribution of balls in two urns**

Now we distribute successively unlabelled balls in two labelled urns, a configuration of the system (state) X_n after the distribution of the n^{th} ball is the number of balls in the first urn and in the second urn $X_n = (X_n^1, X_n^2)$.

III.2.a. Suppose that the probability of the distribution of the n^{th} ball to urn 1 is p and does not depend on the number of balls already distributed. Explain why X_n is a homogeneous Markov chain. Draw the transition graph of the chain. The probability law of X_n is well known, what is it ?

III.2.b. If the probability of the distribution of the n^{th} ball to urn 1 is p_n and depends on the number of balls already distributed then X_n is still a Markov chain but it is not homogeneous. Suppose now that the probability p_n depends on the number of balls in the first urn. If the number of balls in the first urn is a and in the second urn b (n is $a + b + 1$), then $p_n = \frac{a}{a+b}$ (p_n is proportional with the number of balls in urn 1). Starting with an initial state $X_0 = (1, 1)$ compute the probability law of $X_1, X_2,$ and X_3. Compute the probability law of X_n.

III.3. **Probabilistic evolution of an urn dynamic**

Now we suppose that there are exactly N unlabelled balls in two labelled urns. We pick one ball uniformly among the N balls and move it from its urn to the other one. Denote by X_n the number of balls in the first urn after the n^{th} movement. X_n is clearly a Markov chain, compute its stationary distribution (explanations, pictures, curves are welcomed). What do you think about this result ?