Fundamental Computer Science

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Non-deterministic Turing Machine

A Non-deterministic Turing Machine (M) is a sixtuple $(K, \Sigma, \Gamma, \Delta, s, H)$, where K, Σ, Γ, s and H are as in the definition of the Deterministic Turing Machine, and Δ describes the transitions and it is a *subset* of

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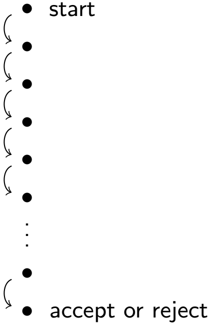
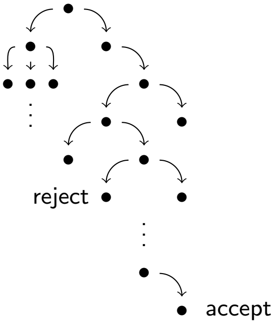
- Δ is not a function
  - a single pair of (q, σ) can lead to multiple pairs (q', σ')
  - the empty string ϵ is allowed as a transition symbol
- A configuration may *yield* several configurations in a single step
  - $\vdash_M$ is not necessarily uniquely identified

Non-determinism

- the next step is **not unique**



deterministic computation



Comparison deterministic vs non-deterministic

Non-deterministic Turing Machine

Definitions

Let $M = (K, \Sigma, \Gamma, \Delta, s, H)$ be a Non-deterministic Turing Machine.
We say that M **accepts** an input $w \in \Sigma^*$ if

$$(s, \underline{\sqcup}w) \vdash_M^* (h, u\underline{\sigma}v)$$

for some $h \in H$, $\sigma \in \Sigma$ and $u, v \in \Sigma^*$.

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We say that M **decides** a language L if for each $w \in \Sigma^*$ the following two conditions hold:

1. there is natural number $N \in \mathbb{N}$ (depending on M and $|w|$) such that there is no configuration c satisfying $(s, \underline{\sqcup}w) \vdash_M^N c$
2. $w \in L$ if and only if $(s, \underline{\sqcup}w) \vdash_M^* (h, u\underline{\sigma}v)$ for some $\sigma \in \Sigma$ and $u, v \in \Sigma^*$

Non-deterministic Turing Machine

Definitions (cont'd)

Let $M = (K, \Sigma, \Gamma, \Delta, s, H)$ be a Non-deterministic Turing Machine.

We say that M **computes** a function $f : \Sigma^* \to \Sigma^*$ if for each $w \in \Sigma^*$ the following two conditions hold:

- $(s, \underline{\sqcup}w) \vdash^*_M (h, \underline{\sqcup}v)$ if and only if $v = f(w)$

Example

- A natural number $m \in \mathbb{N}$ is called *composite* if it can be written as the product of two natural numbers $p, q \in \mathbb{N}$, i.e., $m = p \cdot q$ Describe (high-level) a Non-deterministic Turing Machine that recognizes the language $L = \{1^m : m \text{ is a composite number}\}$.

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- How to transform the above machine to decide the same language?
  1. choose two integers $p < m$ and $q < m$ **non-deterministically**
  2. multiply p and q
  3. compare a with $p \cdot q$ and if they are equal then *accept*, else *reject*

Exercise

- Consider a set $A = \{a_1, a_2, \ldots, a_n\}$ of positive integers and an integer $w \in \mathbb{N}$.
  Give a Non-deterministic Turing Machine that *recognizes* the language $L = \{A' \subseteq A : \sum_{a_i \in A'} a_i = w\}$.

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- How to choose A' non-deterministically?
  - produce all binary numbers of n digits
  - start from $00\ldots0$ and add 1 at each iteration

Non-deterministic Turing Machine

Theorem

Every Non-deterministic Turing Machine $NDTM = (K, \Sigma, \Gamma, \Delta, s, H)$ *has an equivalent Deterministic Turing Machine* DTM*.*

Proof (sketch):

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Proof (sketch):

- Use a multiple tape deterministic Turing Machine

tape 1: input (never changes)
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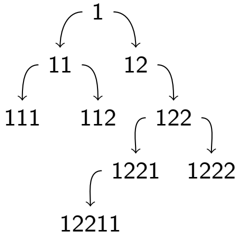
- data on tape 3:
  - each node of the computation tree of $NDTM$ has at most c children
  - address of a node in $\{1, 2, \ldots, c\}^*$



Non-deterministic Turing Machine

Proof (sketch):

1. Initialize tape 1 with the input w and tapes 2 & 3 to be empty.
2. Copy the contents of tape 1 to tape 2.
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  - we need exponential time of steps with respect to NDTM!

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Discussion

- Non-deterministic Turing Machines seem to be more powerful than deterministic ones
- we pay this in computation time

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- next lectures: we will see what does this mean